2 Equations of Motion

In this section, we will derive the six full equations of motion in a non-rotating, Cartesian coordinate system.

2.1 Six equations of motion (non-rotating, Cartesian coordinates)

For reference, the six non-rotating equations of fluid motion in Cartesian coordinates. There are 5 \textit{prognostic} equations, i.e. provide information on how the fluid will evolve, and 1 \textit{diagnostic} equation, i.e. describing the instantaneous state of the fluid.

\textbf{Continuity Equation; Mass Conservation (1)}:

\[
\frac{D\rho}{Dt} + \rho (\nabla \cdot v) = 0 \quad \text{(2.1)}
\]

or

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0 \quad \text{(2.3)}
\]

\textbf{Momentum (Navier-Stokes) Equations; Momentum Conservation (3)}:

\[
\frac{Dv}{Dt} = -\nabla \frac{p}{\rho} - \nabla \Phi + \nu \nabla^2 v \quad \text{(2.4)}
\]

\textbf{Thermodynamic Equation; Energy Conservation (1)}:

\[
\frac{D\theta}{Dt} = \frac{\theta}{T} \frac{Q}{c_p}, \text{ where } \theta = T \left( \frac{p_0}{p} \right)^{R/c_p} \quad \text{(2.5)}
\]

\textbf{Equation of State (1)}:

\[
p = f(\theta, \rho), \text{ e.g. } p = \rho RT \quad \text{(2.6)}
\]

2.2 Continuity (conservation of mass)

Mass is absolutely conserved in classical mechanics, and so, the conservation of mass is arguably our first building block in describing the evolution of a fluid.

2.2.1 \textbf{an Eulerian perspective}

Consider an infinitesimal cube control volume \(\Delta V = \Delta x \Delta y \Delta z\). Fluid moves into and out of the cube at its faces.
The accumulation of fluid within the volume due to flow in the x-direction can be written as
\[ \Delta y \Delta z \left[ (\rho u)(x, y, z) - (\rho u)(x + \Delta x, y, z) \right] = -\frac{\partial (\rho u)}{\partial x} |_{x,y,z} \Delta x \Delta y \Delta z. \] (2.7)

Similarly, in the y-direction
\[ -\frac{\partial (\rho v)}{\partial y} \Delta x \Delta y \Delta z \] (2.8)
and in the z-direction
\[ -\frac{\partial (\rho w)}{\partial z} \Delta x \Delta y \Delta z. \] (2.9)

Note the minus signs - this is because having more leave the right face compared to the left face (a positive derivative) would imply a mass divergence, or, mass leaving the cube. Thus, the minus sign is the accumulation within the control volume.

Thus, the total net accumulation of mass within the control volume can be written as
\[ -\Delta V \left( \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} \right) \] (2.10)

In order to have mass conservation (something we are assuming here, that is, it does not fall out of the equations themselves) the net accumulation must be balanced by a corresponding increase of fluid mass within the control volume. That is,
\[ \frac{\partial (\Delta \rho)}{\partial t} = \Delta V \frac{\partial \rho}{\partial t} = -\Delta V \left( \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} \right) \] (2.11)

using the fact that the volume of the cube does not change in time. Moving all of the terms to the left hand side,
\[ \Delta V \left( \frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} \right) = 0 \] (2.12)

Because the control volume is arbitrary, the quantity in the parentheses must be zero, and thus, the continuity equation is
\[ \frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0. \] (2.13)
2.2.2 a Lagrangian perspective

One could perform a comparable derivation in terms of a fixed fluid parcel to derive the continuity equation from a Lagrangian perspective, but we will not do that here. However, we take a moment to derive the advective/Lagrangian form of continuity starting from 2.13. Using our vector formulas, we know that

\[ \nabla \cdot (\psi \mathbf{a}) = \psi \nabla \cdot \mathbf{a} + \mathbf{a} \cdot \nabla \psi. \quad (2.14) \]

Thus,

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = \frac{\partial \rho}{\partial t} + \rho \nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla \rho = 0 \quad (2.15) \]

Using the definition of the material derivative (1.11), this leads to the Lagrangian form of mass continuity:

\[ \frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v} = 0. \quad (2.16) \]

Note that if the flow is incompressible, then \( \rho = \text{constant} \) and starting from 2.15, the continuity equation becomes

\[ \frac{\partial \rho}{\partial t} + \rho \nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla \rho = 0 \quad (2.17) \]

\[ 0 + \rho \nabla \cdot \mathbf{v} + 0 = 0 \quad (2.18) \]

and since \( \rho \) is constant

\[ \nabla \cdot \mathbf{v} = 0 \quad (2.20) \]

Note that the first numerical weather forecast was of the (seemingly simple) equation:

\[ \frac{D(f + \zeta)}{Dt} = 0. \]

2.3 Momentum equations

The momentum equations describe how the momentum (or velocity \( \times \) mass) evolves in time in response to both internal and imposed forces. Throughout this course, we will be mostly focused on imposed forces from outside of the fluid, rather than internal forces such as viscosity. Depending on the fluid of interest, there can be a wide range of forces acting on a fluid parcel.

The basic starting point is Newton’s 2nd Law: the acceleration of an object depends on the force acting on the object divided by its mass. Or, put another way,

\[ \mathbf{F} = m\mathbf{a}. \quad (2.21) \]
From this, we can solve for the acceleration $a$ of the fluid element as a function of $f$ which is a force per unit volume:

$$\frac{F}{m} = \frac{F}{\rho V} = \frac{f}{\rho} = a$$

(2.22)

However, the acceleration of the parcel is by definition the material derivative of the velocity of the parcel, and so:

$$\frac{F}{m} = \frac{f}{\rho} = a = \frac{Dv}{Dt} = \partial_t v + (v \cdot \nabla)v$$

(2.23)

The goal of deriving the momentum equations (Navier-Stokes equations) is to quantify all of the forces $f$ acting on the parcel. Here, we will describe the dominant forces acting on earth’s atmosphere and ocean: pressure and gravitational forces with a short comment on internal viscous forces. That is,

$$\frac{Dv}{Dt} = \partial_t v + (v \cdot \nabla)v = \frac{F_g}{m} + \frac{F_p}{m} + \frac{F_v}{m}$$

(2.24)

### 2.3.1 Gravity

Newton’s law of gravity states that:

$$F_g = -\frac{GMm}{r^2}r$$

(2.25)

where the term on the far right is there to provide a direction pointing between the centers of mass of one object with mass $M$ and a second object with mass $m$. $G$ is the gravitational constant. For gravity on earth, $M$ is the mass of the earth, and $r = a + z$ where $a$ is the radius of the earth. This results in an acceleration of mass $m$ due to gravity $a_g$ of

$$\frac{F_g}{m} = \frac{f_g}{\rho} = -g_0 \frac{1}{(1 + z/a)^2}\hat{k} \equiv -g\hat{k} \approx -g_0\hat{k}$$

(2.26)

where $g_0 = GM/(a^2) \approx 9.81 \text{ m/s}^2$, and $g = g_0/(1 + z/a)^2$. Note that we have assumed that $z \ll a$.

It turns out it will be useful to define a quantity $\Phi = -GM/r = -GM/(a + z)$ called the gravitational potential, which allows us to write $-g\hat{k}$ without the $\hat{k}$ notation:

$$-g\hat{k} = -\nabla \Phi.$$  

(2.27)

Since $\nabla \Phi$ is the gradient of $\Phi$, in Cartesian coordinates,

$$-\nabla \Phi = \frac{\partial \Phi}{\partial x} \hat{i} + \frac{\partial \Phi}{\partial y} \hat{j} + \frac{\partial \Phi}{\partial z} \hat{k}$$

(2.28)
From the definition of $\Phi$ we know that it is only a function of $z$, and thus, only changes in the local vertical direction. Thus,

$$-\nabla \Phi = 0 + 0 + \frac{\partial}{\partial z} \left( \frac{-GM}{(a + z)} \right) \hat{k} \quad (2.29)$$

$$= -GM \frac{\partial}{\partial z} \left( \frac{1}{(a + z)} \right) \hat{k} \quad (2.30)$$

$$= -GM \left( \frac{1}{a + z} \right)^2 \hat{k} \quad (2.31)$$

$$= -\frac{GM}{a^2 (1 + z/a)^2} \hat{k} \quad (2.32)$$

$$= -g \hat{k} \quad (2.33)$$

Thus, the acceleration due to the gravitational force is

$$\frac{F_g}{m} = -\nabla \Phi. \quad (2.34)$$

We have our first acceleration to add to the right-hand-side of our momentum equation:

$$\begin{align*}
\frac{Dv}{Dt} &= -\nabla \Phi + \ldots \quad (2.35)
\end{align*}$$

### 2.3.2 Pressure gradient force

Pressure $p$ is a scalar field defined as the force per unit area ($\text{A}$). Pressure always acts normal to the face of the area (denoted $\hat{n}$), that is, it always acts in the direction that makes a 90 degree angle with the surface of the area. Thus, the pressure force acting on an area is $F_p = pA\hat{n}$.

Consider a fluid element, say a cube, of volume $\Delta V = \Delta x \Delta y \Delta z$. If the pressure is the same on all sides of the cube, it is probably intuitive to you that the no net force will be felt. Now instead consider the case where the pressure force on the left-hand-side of the cube is

$$F_{LHS} = p\Delta y \Delta z \quad (2.36)$$
and the pressure force on the right-hand-side of the cube is

\[ F_{\text{RHS}} = (p + \Delta p) \Delta y \Delta z \] (2.37)

due to a slightly different pressure of \((p + \Delta p)\). It is the pressure difference, \(\Delta p\) that determines the net force felt on the cube. We can rewrite \(\Delta p = \partial_x p \Delta x\), and write the net force in the x-direction as the sum of these two pressure forces:

\[
\text{net force} = F_{\text{LHS}} - F_{\text{RHS}} = p \Delta y \Delta z - (p + \partial_x p \Delta x) \Delta y \Delta z = -\partial_x p \Delta x \Delta y \Delta z = -\partial_x p \Delta V
\] (2.38)

where we used the fact that the pressure force is into the face, and thus, negative for the right-hand-side.

A similar argument applies for the pressure force in the y-direction and the z-direction. That is

\[
F_{px} = -\partial_x p \Delta V
\] (2.39)
\[
F_{py} = -\partial_y p \Delta V
\] (2.40)
\[
F_{pz} = -\partial_z p \Delta V
\] (2.41)

Using vector notation, the net force on the fluid element due to pressure differences at its boundaries is

\[
\mathbf{F}_p = -\Delta V (\partial_x \mathbf{i} \Delta y \Delta z + \partial_y \mathbf{j} \Delta x \Delta z + \partial_z \mathbf{k} \Delta x \Delta y) = -\Delta V \nabla p
\] (2.42)

Thus, the pressure force is

\[
\mathbf{F}_p = -\Delta V \nabla p
\] (2.43)

and the acceleration due to this force is just the force divided by the mass which yields

\[
\frac{\mathbf{F}_p}{m} = -\frac{\Delta V}{m} \nabla p = -\frac{\nabla p}{\rho}
\] (2.44)

Adding this to our momentum equation:

\[
\frac{D\mathbf{v}}{Dt} = -\nabla \Phi + \frac{\nabla p}{\rho} + ...
\] (2.45)

### 2.3.3 Diffusion/Viscosity (internal)

Viscosity results from the internal friction in the fluid from the different molecules interacting with one another. This internal friction is essential for dissipating energy at the molecular scale. Note that this is not the same as the frictional forces acting at the boundary of the earth’s surface. A formal derivation of viscous effects will not be given here, however, viscous forces are often simply thought of as acting to diffuse momentum, i.e.

\[
\mathbf{f}_v = \mu \nabla^2 \mathbf{v}
\] (2.46)
where $\mu$ is the viscosity. To turn $f_v$ into a force per unit density ($f_v/\rho$), one can define the *kinematic viscosity* $\nu \equiv \mu/\rho$. However, for large-scale atmospheric and oceanic fluid motions, viscous effects can be neglected to good approximation.

The symbol $\nabla^2$ is just what it looks like, instead of taking the first-order derivatives of the field, ones takes the second-order derivatives but in a special way. More specifically, $\nabla^2 = \nabla \cdot \nabla()$, that is, it is the *divergence of the gradient* of the field. This operator is known as the *Laplacian*, and it can describe how materials (or heat) concentrations change in time using Fick’s first and second laws:

- **Fick’s 1st Law**: the flux of a material $M$ goes from regions of high to low concentration with a magnitude proportional to the concentration gradient (you can think of this like an advective flux due to molecular motions)

  $$\text{Flux} \propto \nabla M$$ (2.47)

- **Fick’s 2nd Law**: diffusion causes the concentration of a material $M$ to change with time in a manner such that the rate of change of $M$ is proportional to the divergence of the flux

  $$\frac{\partial M}{\partial t} \propto \nabla \cdot \nabla M = \nabla^2 M$$ (2.48)

### 2.3.4 Putting the pieces together

We have finally come to the point where we can combined our accelerations due to gravitational, pressure and viscous forces into a set of three momentum equations (one for each direction), or one vector momentum equation.

$$\frac{Dv}{Dt} = \partial_t v + (v \cdot \nabla) v = \frac{F_g}{m} + \frac{F_p}{m} + \frac{F_v}{m}$$ (2.49)

$$= -\nabla \Phi + \frac{\nabla p}{\rho} + \nu \nabla^2 v$$ (2.50)

If one removes viscosity, we obtain the *Euler equations*.

$$\frac{Dv}{Dt} = \partial_t v + (v \cdot \nabla) v = -\nabla \Phi + \frac{\nabla p}{\rho}$$ (2.51)

For each of the 3-direction individually, the momentum equations are

$$\frac{Du}{Dt} = \partial_t u + (v \cdot \nabla) u = \frac{1}{\rho} \frac{\partial p}{\partial x}$$ (2.52)

$$\frac{Dv}{Dt} = \partial_t v + (v \cdot \nabla) v = \frac{1}{\rho} \frac{\partial p}{\partial y}$$ (2.53)

$$\frac{Dw}{Dt} = \partial_t w + (v \cdot \nabla) w = \frac{1}{\rho} \frac{\partial p}{\partial z} - g$$ (2.54)
where recall that the advection term expands to be $(v \cdot \nabla) = u \partial_x + v \partial_y + w \partial_z$.

Thus, expanding everything out for $u$ as an example leads to:

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + v \cdot \nabla u = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = \frac{1}{\rho} \frac{\partial p}{\partial x} \tag{2.55} $$

Note that there are 3 momentum equations, but 5 unknowns ($u, v, w, p, \rho$). The continuity equation provides 1 more equation, but we still have only 4 equations for 5 unknowns. We therefore need at least 1 more equation to have a complete set.

### 2.4 Equation of state

An equation of state is a *diagnostic* equation - that is, it relates the variables to one another but does not provide information about their evolution in time. The conventional equation of state is one that relates pressure, temperature, composition and density.

You are likely very familiar with the *ideal gas law*, which is the equation of state for an ideal gas:

$$p = \rho RT, \tag{2.56}$$

where $R$ is the *specific gas constant* and $R_{air} = 287 \text{ J/kg/K}$. It turns out that the atmosphere can be very well approximated as an ideal gas, and so, this equation will be useful over and over again. We will now briefly discuss where this equation comes from.

"Derive" from basic understanding of kinetic gas theory.

In an ideal gas, the molecules are treated as point masses that only interact with each other through elastic collisions. Kinetic gas theory tells us that the macroscopic pressure, $p$, and temperature, $T$, are related to the average kinetic energy of the molecules:

$$p = \frac{2}{3} n \left\langle \frac{m}{2} v^2 \right\rangle \tag{2.57}$$

$$T = \frac{2}{3k_B} \left\langle \frac{m}{2} v^2 \right\rangle \tag{2.58}$$

$$\Rightarrow p = nk_B T \tag{2.59}$$

where $n$ is the molecular number density, $m$ is the mass of each molecule and $k_B = 1.38 \times 10^{-23} \text{ J/K}$ is the Boltzmann constant. By definition, the density is given by $\rho = n m$ (# of molecules $\times$ mass of each molecule / volume).

Putting things together:

$$p = \rho \frac{k_B}{m} T = \rho RT \tag{2.60}$$
where $R \equiv \frac{k_m}{m}$ (the specific gas constant), and $R_{air} = 287$ J/kg/K. Thus we obtain the equation of state for an ideal gas, namely, “the ideal gas law”. This is a diagnostic that relates pressure, density and temperature.

We now have an issue though, although we’ve added an equation to our equation set, we’ve also introduced another variable, namely, $T$! Thus, we require yet another equation - the 1st law of thermodynamics (i.e. energy conservation), which we will discuss in the next section.

### 2.4.1 Barotropic vs. baroclinic fluids

A fluid in which the pressure is only a function of density, $p = p(\rho)$, is called a **barotropic fluid**. In this special case, density (or pressure) can be eliminated from the set of equations and the system is closed with five equations and five unknowns (note that temperature doesn’t show up in our barotropic system).

In the more general case, so-called **baroclinic case**, pressure is a function of at least (potential) temperature as well (as in the ideal gas law). Thus, $p = p(\rho, T)$ or $p = p(\rho, \theta)$.

### 2.4.2 Seawater

Now is the moment where many of you learn that you have taken the ideal gas law for granted all of these years. It may not come as a surprise that seawater is not an ideal gas. However, what may be surprising is that no simple expression like the ideal gas law exists for sea water, or liquids in general! For seawater specifically, the equation of state takes the general form $p = p(\rho, T, S)$, where $S$ is the salinity of the water.

Taking advantage of the fact that density variations are generally small in the ocean, one can derive a simple approximate equation of state by linearizing about a constant background density $\rho_0$ (~1000 kg/m$^3$).

Let $\rho = \rho(\rho_0, T, S, p)$.

$$\rho = \rho_0 + \delta \rho = \rho_0 + (\partial_\rho \rho)|_{p,s,T} \delta T + (\partial_\rho \rho)|_{p,T} \delta S + (\partial_\rho \rho)|_s \delta p$$

$$= \rho_0 \left[ 1 + \frac{1}{\rho_0} (\partial_\rho \rho)|_{p,s} \delta T + \frac{1}{\rho_0} (\partial_\rho \rho)|_{p,T} \delta S + \frac{1}{\rho_0} (\partial_\rho \rho)|_s \delta p \right]$$

$$= \rho_0 \left[ 1 - \beta_T \delta T + \beta_S \delta S + \frac{1}{\rho_0 c_s^2} \delta p \right]$$

where

- $\beta_T$: thermal expansion coefficient (\sim $2 \cdot 10^{-4}$ 1/K)
- $\beta_S$: haline contraction coefficient (\sim $7 \cdot 10^{-4}$ 1/ppt, ppt = parts per thousand)
- $c_s$: speed of sound for seawater (\sim 1500 m/s)

Defining reference-state temperature ($T_0$), salinity ($S_0$) and pressure ($p_0$) constants leads to

$$\rho = \rho_0 + \delta \rho \approx \rho_0 \left[ 1 - \beta_T (T - T_0) + \beta_S (S - S_0) + \frac{1}{\rho_0 c_s^2} (p - p_0) \right]$$
Typical variations of density, salinity, temperature and pressure are in the ocean are:

- density: 1020-1040 kg/m$^3$
- salinity: 23-36 ppt
- temperature: 0-25°C
- pressure: $1.0 \times 10^4 - 1.0 \times 10^8$ kg/(m·s$^2$), hPa

**Point of Discussion:**

If you plug-in the variations of salinity, temperature and pressure, which term makes the largest impact on density?

Salinity:
$$\rho_0 \left[ 1 - 2 \cdot 10^{-4} (0 - 25) \right] \approx 1.005 \rho_0$$

Salinity:
$$\rho_0 \left[ 1 + 7 \cdot 10^{-4} (26 - 23) \right] \approx 1.002 \rho_0$$

Pressure:
$$\rho_0 \left[ 1 + \frac{1}{1000-1500^2} (1.0 \times 10^8 - 1.0 \times 10^4) \right] \approx 1.05 \rho_0$$

The answer is clearly pressure, meaning the pressure is the dominant contributor to density changes in the ocean. However, also note that even with the largest changes we can apply, density only varies by 5%. Thus, seawater is nearly incompressible.

Since the equation of state for seawater introduces yet another new quantity on top of temperature - salinity - we need another equation for salinity, specifically, a prognostic equation:

$$\frac{DS}{Dt} = \begin{cases} 
\frac{S}{h} (E - P) & \text{surface mixed layer} \\
0 & \text{ocean interior} 
\end{cases}$$

(2.65)

where $E - P$ denotes evaporation minus precipitation, and $h$ is the thickness of the mixed layer. This equation is another continuity equation, namely, *salinity conservation*.

If you are further interested in calculating particular attributes of seawater (e.g. the speed of sound, or the potential temperature), the following website might be of interest: [http://fermi.jhuapl.edu/denscalc.html](http://fermi.jhuapl.edu/denscalc.html).

### 2.5 Thermodynamic equation

In general, the equation of state introduces another thermodynamic variable. In the case of the atmosphere, this was true, namely, temperature was introduced in the ideal gas law (equation of state). We therefore need another equation relating the thermodynamic properties of the fluid (pressure, density, temperature, and possibly salinity in the case of seawater), to close the system. This equation will be derived
from energy conservation, more specifically, from the 1st law of thermodynamics. In general, we assume local thermodynamic equilibrium.

We begin by defining the internal energy per unit mass of a system:

\[ I = I(\alpha, \eta) \quad \text{or} \quad I = I(\alpha, T) \]  \hspace{1cm} (2.66)

where we are ignoring the third variable \( S \) that corresponds to composition, and where \( \alpha = 1/\rho \) is the specific volume, \( \eta \) is the entropy and \( T \) is the temperature.

- **entropy**: a measure of the number of specific ways in which a thermodynamic system may be arranged, commonly understood as a measure of disorder

Then, the differential of this equation gives us changes in the internal energy as a function of \( \alpha \) and \( \eta \):

\[ dI = \left( \frac{\partial I}{\partial \eta} \right) |_{\alpha} d\eta + \left( \frac{\partial I}{\partial \alpha} \right) |_{\eta} d\alpha \]  \hspace{1cm} (2.67)

Note that the entropy is a function of temperature and \( \eta \), and thus, the two can be interchanged in this context.

The 1st law of thermodynamics states that a change in internal energy of a fluid \((dI)\) is given by the difference of heat input \((dQ)\) and work done by the pressure force \((dW)\):

\[ dI = dQ - dW \]  \hspace{1cm} (2.68)

We now define temperature and pressure in terms of differentials of internal energy in order to rewrite the 1st law of thermodynamics in more physical terms and link 2.67 with 2.68.

- \( p = -\left( \frac{\partial I}{\partial \alpha} \right) |_{\eta} \)

- \( T = \left( \frac{\partial I}{\partial \eta} \right) |_{\alpha} \)

These may be considered the defining relations for these variables, and note that they jive with our physical intuition of these properties of fluid molecules.

Using these relations, we can rewrite the 2.67 as

\[ dI = T d\eta - p d\alpha \]  \hspace{1cm} (2.69)

Moreover

- **Work done**: the work done by a body is equal to the pressure times the change in volume, and thus, per unit mass

\[ dW = p d\alpha \]  \hspace{1cm} (2.70)
• **Heat input:** the change in entropy associated with a heat input into a body (assuming a *reversible process*) is

\[ dQ = Td\eta \] (2.71)

Putting everything together, 2.69 becomes

\[
dI = Td\eta - pd\alpha = dQ - dW
\] (2.72)

which brings us back to the 1st law of thermodynamics.

An important note is that \( Q \) and \( W \) are not functions of the state of the body...they are only defined as (imperfect) differentials. They should only be thought of in terms of fluxes - there sum changes the internal energy of the body.

### 2.5.1 An ideal gas: the atmosphere

In an ideal gas, the internal energy is described by the kinetic energy of the molecules, and thus, is only a function of temperature: \( I = I(T) \). A *simple ideal gas* is one where the internal energy is a linear function of temperature, that is, \( I = cT \), where \( c \) is a constant associated with the particular ideal gas of interest. The constant \( c \) is called the *heat capacity* of the gas. We can determine the constant heat capacity of a simple gas in the following way:

Assuming the composition of the fluid is constant, then we know from the 1st law of thermodynamics 2.69 and 2.71 that

\[ dQ = dI + pd\alpha \] (2.73)

and that this is always true (no approximations about an ideal gas have been made, although we have assumed the process is reversible).

In general, \( I = I(\alpha, \eta) \) or \( I = I(\alpha, T) \), so, \( dI \) can be expanded as

\[
dI = \frac{\partial I}{\partial T}|_{\alpha}dT + \frac{\partial I}{\partial \alpha}|_{T}d\alpha
\] (2.74)

Thus,

\[
dQ = \frac{\partial I}{\partial T}|_{\alpha}dT + \frac{\partial I}{\partial \alpha}|_{T}d\alpha + pd\alpha
\] (2.75)

\[
dQ = \frac{\partial I}{\partial T}|_{\alpha}dT + \left( \frac{\partial I}{\partial \alpha}|_{T} + p \right) d\alpha
\] (2.76)
Thus, if we have a simple ideal gas such that $I = cT$, $d\alpha = 0$ (constant volume) and we get the special case of

$$dQ = \frac{\partial I}{\partial T}|_{\alpha}dT$$

(2.77)

Thus,

$$c_v \equiv \frac{\partial Q}{\partial T}|_{\alpha} = \frac{\partial I}{\partial T}|_{\alpha}$$

(2.78)

where we use $c_v$ to specify the specific heat at constant volume (constant $\alpha$). That is, the constant $c$ in $I = cT$ is $c_v$.

We can also determine the heat capacity at constant pressure $c_p$:

$$dQ = dI + pd\alpha$$

(2.79)

$$= d(I + p\alpha) - \alpha dp$$

(2.80)

$$= dh - \alpha dp$$

(2.81)

$$= \frac{\partial h}{\partial T}|_p dT + \frac{\partial h}{\partial p}|_T dp - \alpha dp$$

(2.82)

$$= \frac{\partial h}{\partial T}|_p dT + \left( \frac{\partial h}{\partial p}|_T - \alpha \right) dp$$

(2.83)

$$= \frac{\partial h}{\partial T}|_p dT + \left( \frac{\partial h}{\partial T}|_p - \alpha \right) dp$$

(2.84)

where $h \equiv I + p\alpha$, and has the special name of enthalpy. From this we can read-off that

$$c_p \equiv \frac{\partial Q}{\partial T}|_p = \frac{\partial h}{\partial T}|_p$$

(2.85)

- **enthalpy**: thermodynamic potential of a system - at constant pressure the change in enthalpy is related to a change in internal energy and a change in the volume, which is multiplied by the constant pressure of the system.

Note that in 2.85 we never made an assumption of an ideal gas, and thus, the equation holds generally. For the special case of an ideal gas,

$$p\alpha = RT$$

(2.86)

and so

$$h = I + RT = c_v T + RT = T(c_v + R).$$

(2.87)

Starting from the definition of $c_p$ (2.85)

$$c_p = \frac{\partial h}{\partial T}|_p = c_v + R$$

(2.88)
and therefore
\[ c_p = c_v + R \] (2.89)

Putting everything from this section together, we can arrive at the thermodynamic equation for an ideal gas (using \( dI = c_v dT \)):

\[ dQ = c_v dT + p d\alpha \] (2.90)

and taking the total time derivative of both sides leads to one version of the thermodynamic equation

\[ \dot{Q} = c_v \frac{DT}{Dt} + p \frac{D\alpha}{Dt} \] (2.91)

or from 2.83 and 2.87

\[ dQ = c_p dT - \alpha dp \] (2.92)

and taking the total time derivative of both sides leads to another version of the thermodynamic equation

\[ \dot{Q} = c_p \frac{DT}{Dt} - \alpha \frac{Dp}{Dt} \] (2.93)

### 2.5.2 Potential temperature, adiabatic processes

Using the ideal gas law in the form of \( \alpha = RT/p \), 2.93 can be rewritten as

\[ \dot{Q} = c_p \frac{DT}{Dt} - \frac{RT}{p} \frac{Dp}{Dt} \Rightarrow \frac{\dot{Q}}{c_p T} = \frac{D}{Dt} \ln \left( \frac{T}{p^\kappa} \right) - \frac{R}{c_p} \frac{D}{Dt} \ln p \] (2.94)

recalling that

\[ \frac{d}{dt} \ln X = \frac{1}{X} \frac{dX}{dt} \] (2.95)

and therefore,

\[ \frac{\dot{Q}}{c_p T} = \frac{D}{Dt} \ln \left( \frac{T}{p^\kappa} \right) \] (2.96)

where \( \kappa \equiv R/c_p \).

- **Adiabatic process**: a process where no heat is exchanged, that is, \( \dot{Q} = 0 \).

It follows then for an *adiabatic process* that the quantity \( T/p^\kappa \) is materially conserved:

\[ \frac{D}{Dt} \ln \left( \frac{T}{p^\kappa} \right) = 0 \Rightarrow \frac{D}{Dt} \left( \frac{T}{p^\kappa} \right) = 0 \] (2.97)
Put another way, if \( T_0 \) and \( p_0 \) denote reference values of temperature and pressure respectively, then

\[
\frac{T}{p^\kappa} = \frac{T_0}{p_0^\kappa} \implies \frac{T}{T_0} = \left(\frac{p}{p_0}\right)^\kappa
\]  

(2.98)

From this, we define the potential temperature \( \Theta \):

\[
\Theta \equiv T_0 = T \left(\frac{p_0}{p}\right)^\kappa
\]  

(2.99)

where \( p_0 \) is typically set to 1000 hPa (~mean sea level pressure).

From its definition, potential temperature is materially conserved for adiabatic processes: \( \frac{D\Theta}{Dt}\big|_{\text{adiab}} = 0 \).

- **potential temperature:** the temperature an air-parcel would obtain if adiabatically compressed to 1000 hPa.

Show plot of potential temperature - discuss vertical derivative.

Using potential temperature, we can rewrite the thermodynamic equation for an ideal gas as

\[
\frac{D\Theta}{Dt} = \frac{\theta}{T_c} \frac{\dot{Q}}{c_p}
\]  

(2.100)

and this is the version that will be used most often in this class.

For an ideal gas, one can derive a simple equation between the potential temperature and entropy using the knowledge that: \( T d\eta = c_v dT + pd\alpha = c_p dT - \alpha dp \) (recall that \( p\alpha = RT \) and \( c_p = c_v + R \)):

\[
d\eta = c_p \frac{dT}{T} - \frac{1}{\rho T} dp = c_p \cdot d(\ln T) - R \cdot d(\ln p)
\]

\[
= c_p \cdot (d(\ln T) - \kappa d(\ln p)) = c_p d(\ln \frac{T}{p^\kappa}) = c_p d \ln \left(\frac{\Theta}{p_0^\kappa}\right)
\]  

(2.101)

(2.102)

Thus, if the process is adiabatic, \( \Theta \) is constant and \( \eta \), the entropy, is also constant! Thus, surfaces of constant potential temperature are often referred to as *isentropic surfaces* or simply *isentropes*.

We have now reached our first big conclusion in this course, we have a set of 6 equations (5 prognostic: 3 momentum, 1 continuity and 1 thermodynamic; 1 diagnostic: 1 equation of state) and 6 unknowns (\( u, v, w, p, \rho, \Theta \) or \( T \)).

**Continuity Equation; Mass Conservation (1):**

\[
\frac{D\rho}{Dt} + \rho (\nabla \cdot v) = 0
\]  

(2.103)

\[
\text{or}
\]

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0
\]  

(2.104)

(2.105)
Momentum (Navier-Stokes) Equations; Momentum Conservation (3):

\[
\frac{\text{D} \mathbf{v}}{\text{D} t} = -\nabla \frac{p}{\rho} - \nabla \Phi + \nu \nabla^2 \mathbf{v} \tag{2.106}
\]

Thermodynamic Equation; Energy Conservation (1):

\[
\frac{\text{D} \theta}{\text{D} t} = \frac{\theta}{T} \frac{\dot{Q}}{c_p}, \text{ where } \theta = T \left( \frac{p_0}{p} \right)^{R/c_p} \tag{2.107}
\]

Equation of State (1):

\[
p = f(\theta, \rho), \text{ e.g. } p = \rho RT \tag{2.108}
\]