10 Shallow Water Models

So far, we have studied the effects due to rotation and stratification in isolation. We then looked at the effects of rotation in a barotropic model, but what about if we add stratification as well? To do this, we will introduce "shallow water models", a step-up in complexity from the 2-d non-divergent barotropic vorticity equation. Shallow water models allow for a combined analysis of both rotation and stratification in a simplified system.

As discussed in Chapter 3 of Vallis, although some of the simplifications we will make for the shallow water equations may seem unrealistic (e.g. constant density fluid with solid bottom boundary), they are actually still incredibly useful tools for understanding the atmosphere and ocean. By adding layers, we can subsequently study the effects of stratification - and visualize the atmosphere being made of many layers of near constant density fluid, with each layer denoted by different potential temperatures. For the ocean, each layer is denoted by different isopycnals.

10.1 Single-layer shallow water equations

Consider a single layer of fluid bounded below by a rigid surface (cannot move/deform) and above by a free surface (can move/deform), as depicted in Figure 3.1 of Vallis (see below).

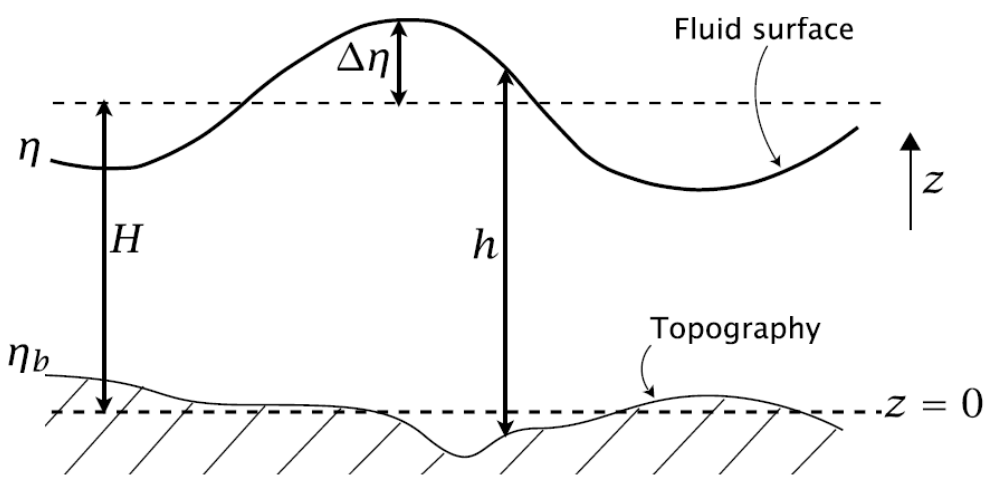

Fig. 3.1 A shallow water system. h is the thickness of a water column, H its mean thickness, η the height of the free surface and η_b is the height of the lower, rigid, surface, above some arbitrary origin, typically chosen such that the average of η_b is zero. $\Delta\eta$ is the deviation free surface height, so we have $\eta = \eta_b + h = H + \Delta\eta$.

Why is this system called the *shallow water* equations?

- **shallow:** suggests a small aspect ratio, $H/L << 1$, i.e. the use of the hydrostatic approximation

- **water:** suggests a fluid of constant density ρ_0

Using these simplifications, we obtain for our equations of motion:

- hydrostatic balance, i.e. a diagnostic relation in place of the vertical momentum equation: $\partial_z p = -\rho g$
- non-divergent flow from the continuity equation: $D\rho_0/Dt = 0$
- no thermodynamic equation required, and no equation of state (since ρ_0 is a constant rather than an unknown variable)

10.1.1 Momentum equations

The hydrostatic relation for a fluid of constant density ρ_0 is hydrostatic balance, which can be integrated to give:

$$\frac{\partial p}{\partial z} = -\rho_0 g \qquad \Rightarrow \qquad p(x, y, z) = -\rho_0 g z + c \tag{10.1}$$

where c is a constant. This constant can be determined by noting that the pressure at the *top of the layer* equals some background value $p(x, y, z = \eta) = p_0$. We simply set p_0 to zero, which corresponds to the assumption that the weight of the fluid above our layer is negligible). Thus,

$$c = p_0 + g\rho_0\eta, \qquad \text{hence} \qquad p(x, y, z) = g\rho_0\left[\eta(x, y) - z\right] \tag{10.2}$$

Note that the only contribution to the pressure that is dependent on (x, y) is the one involving $\eta(x, y)$. This greatly simplifies the pressure gradient force in the horizontal equations of motion:

$$-\frac{1}{\rho_0}\frac{\partial p}{\partial x} = -\frac{1}{\rho_0} g\rho_0 \frac{\partial \eta}{\partial x} = -g\frac{\partial \eta}{\partial x} \tag{10.3}$$

and similarly for the pressure gradient in the y-direction.

That is, the pressure gradient force is only a function of x and y. It follows then that if u and v are initially uniform in height, they will remain so forever and in that case we may assume that the horizontal flow has no vertical structure (i.e. $\partial_z u = \partial_z v = 0$). This leads to a two-dimensional material derivative of the horizontal flow:

$$\frac{D(u, v)}{Dt} = \frac{\partial(u, v)}{\partial t} + u\frac{\partial(u, v)}{\partial x} + v\frac{\partial(u, v)}{\partial y} + \cancel{w\frac{\partial(u, v)}{\partial z}} \tag{10.4}$$

Putting all of this together, *the horizontal momentum equations are the same as in two-dimensional flow, but* $w \neq 0$*!* A non-zero w turns out to be critical for shallow water dynamics.

Thus, our horizontal momentum equations become:

$$\boxed{\left.\frac{D\mathbf{u}}{Dt}\right|_H + \mathbf{f} \times \mathbf{u} = -g\nabla_H \eta, \qquad \text{with} \qquad \left.\frac{D}{Dt}\right|_H = \partial_t + u\partial_x + v\partial_y} \tag{10.5}$$

10.1.2 Continuity equation:

Since the fluid is incompressible, the divergence of the 3D flow is zero. Writing this out in component form gives

$$\nabla \cdot \mathbf{v} = 0 \quad \Rightarrow \quad \frac{\partial w}{\partial z} = -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = -\nabla_H \cdot \mathbf{u} \tag{10.6}$$

Integrating this from the bottom of the fluid ($z = \eta_b$) to the top ($z = \eta$) gives

$$w(\eta) - w(\eta_b) = \int_{\eta_b}^{\eta} -\nabla_H \cdot \mathbf{u} dz = -\nabla_H \cdot \mathbf{u} \int_{\eta_b}^{\eta} dz = -h\nabla_H \cdot \mathbf{u} \tag{10.7}$$

where we have used the fact that $\mathbf{u}$ is independent of z and $h(x, y) = \eta - \eta_b$ is the depth of the fluid at position (x, y).

The vertical velocities at the layer's top ($w(\eta)$) and bottom ($w(\eta_b)$) are simply the material derivatives of the vertical position of a fluid elements, i.e.

$$w(\eta) \equiv \frac{D\eta}{Dt}\Big|_H, w(\eta_b) \equiv \frac{D\eta_b}{Dt}\Big|_H \quad \Rightarrow \quad w(\eta) - w(\eta_b) = \frac{D}{Dt}(\eta - \eta_b)\Big|_H = \frac{Dh}{Dt}\Big|_H \tag{10.8}$$

Therefore,

$$\boxed{\frac{Dh}{Dt}\Big|_H + h\nabla_H \cdot \mathbf{u} = 0 \quad \text{or} \quad \frac{\partial h}{\partial t} + \nabla_H \cdot (h\mathbf{u}) = 0} \tag{10.9}$$

10.2 Stretching and vertical velocity

Recall that since the 3D flow is non-divergent,

$$\frac{\partial w}{\partial z} = -\nabla_H \cdot \mathbf{u} \tag{10.10}$$

Since the divergence of the horizontal flow is not a function of z, neither is $\partial_z w$. Thus, we can integrate from the bottom to some height z:

$$\int_{\eta_b}^{z} \frac{\partial w}{\partial z'} dz' = \int_{\eta_b}^{z} -\nabla_H \cdot \mathbf{u} dz' \quad \Rightarrow \quad w(z) = w(\eta_b) - (\nabla_H \cdot \mathbf{u})(z - \eta_b) \tag{10.11}$$

That is, the vertical velocity is a *linear* function of height z. In material form

$$\frac{Dz}{Dt} = \frac{D\eta_b}{Dt}\Big|_H - (\nabla_H \cdot \mathbf{u})(z - \eta_b) \tag{10.12}$$

where we have used the fact that $w(z) \equiv D/Dt(z)$ and $w(\eta_b) \equiv D/Dt(\eta_b)$ at the lower boundary.

The above equations holds at an arbitrary vertical position z, however, we know our upper-boundary condition as well. That is, we know that at $z = \eta$, $w(z = \eta) \equiv D/Dt(\eta)$. Thus, at this upper boundary

$$\frac{D\eta}{Dt} = \frac{D\eta_b}{Dt}\bigg|_H - (\nabla_H \cdot \mathbf{u})(\eta - \eta_b) \tag{10.13}$$

Thus eliminating the divergence of the horizontal winds from the last two equations by subtracting them from each other leads to:

$$\frac{D(z-\eta_b)}{Dt} = \frac{z-\eta_b}{\eta-\eta_b}\frac{D}{Dt}(\eta-\eta_b) \tag{10.14}$$

We now recall quotient rule, which tells us that:

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{g(x)f'(x) - g'(x)f(x)}{g^2(x)} \tag{10.15}$$

Letting $f(x) = z - \eta_b$ and $g(x) = \eta - \eta_b = h$, we can rewrite our equation as

$$\frac{h}{h^2}\frac{D(z-\eta_b)}{Dt} = \frac{z-\eta_b}{h^2}\frac{Dh}{Dt} \quad \Rightarrow \quad \frac{h\frac{D}{Dt}(z-\eta_b) - \frac{D}{Dt}h(z-\eta_b)}{h^2} = 0 \tag{10.16}$$

and therefore,

$$\boxed{\frac{D}{Dt}\left(\frac{z-\eta_b}{\eta-\eta_b}\right) = \frac{D}{Dt}\left(\frac{z-\eta_b}{h}\right) = 0} \tag{10.17}$$

Hence, in a shallow water system, fluid stretches uniformly throughout the column. That is, the ratio of the height of a fluid parcel above the floor to the total depth of the column is fixed. For example, if a fluid parcel starts off initially 1/3rd of the way up the column, then as the column increases and decreases in height, the parcel will always remain 1/3rd of the way up.

10.3 Shallow water waves

We will next explore the shallow water equations in the context of the waves that can propagate within them. We will begin by considering the simplest case with no rotation, and then we will generalize to shallow water waves on the f-plane. Finally, we will consider the $\beta-$plane in the shallow water context and discuss the existence of shallow water Rossby waves which will introduce the concept of potential vorticity.

10.3.1 Shallow water gravity waves: no rotation $(f = 0)$ (Vallis 3.7.1)

As we did for the derivation of Rossby waves, we will begin by linearizing the single-layer shallow water equations around some basic state flow:

$$u = u_0 + u', \qquad v = v_0 + v' \tag{10.18}$$

where u_0 and v_0 are constants. We will assume no bottom topography: η_b = constant and $\nabla\eta = \nabla h = \nabla h'$ where $h = H + h'$ (refer back to Figure 3.1 of Vallis).

Let us first consider the simplest case of a resting basic state ($u_0 = v_0 = 0$) in one spatial dimension, in which case, the linearized non-rotating shallow water equations become:

$$\partial_t u' = -g\partial_x h', \qquad \partial_t h' + H\partial_x u' = 0 \tag{10.19}$$

Taking the x-derivative of the u' equation and the time-derivative of the h' equation and combining the two equations, we obtain a wave equation in h':

$$(\partial_{tt} - gH\partial_{xx})h' = 0 \tag{10.20}$$

In order to obtain the dispersion relation, we once again make the standard plane-wave assumption (note the constant coefficients). That is, we make the ansatz that $h' \sim \exp[i(kx - \omega t)]$. Since $\partial_{tt} \to -\omega^2$ and $\partial_{xx} \to -k^2$ we obtain the following dispersion relation:

$$\boxed{\omega^2 = gHk^2 = c_0^2k^2, \quad \text{where} \quad c_0^2 \equiv gH \quad \Rightarrow \quad \omega = \pm gHk = \pm c_0 k} \tag{10.21}$$

Since we are currently operating in a single-layer shallow water system, these waves travel along the free-surface of the layer. These waves are called *shallow water gravity waves*, as will be clear soon.

The first thing to note from the dispersion relation is that these waves *are non-dispersive* since the frequency ω is a linear function of k, and thus, all waves travel with the phase speed $c_0 = \sqrt{gH}$ in the east or west direction. The group velocity $c_g = \partial_k\omega$ is equal to the phase speed for these waves (since ω is a linear function of k).

Note how the shallow water gravity wave phase speed c_0 is proportional to square root of gravity. That is, *gravity is the restoring force for these waves.* The phase speed is also proportional to the square root of the mean depth H.

It is straightforward to generalize the above wave equation to a non-resting basic state in two dimensions:

$$\frac{D_0\mathbf{u}'}{Dt} = -g\nabla h', \qquad \frac{D_0 h'}{Dt} + H(\partial_x u' + \partial_y v') = 0, \quad \text{where} \quad \frac{D_0}{Dt} = \partial_t + u_0\partial_x + v_0\partial_y \tag{10.22}$$

We form a divergence equation from the momentum equations by applying the del-operator to both sides:

$$\frac{D_0}{Dt}\left(\partial_x u' + \partial_y v'\right) = -g\nabla^2 h' \tag{10.23}$$

Then, we insert $H(\partial_x u' + \partial_y v') = -D_0/Dt(h')$ to yield

$$\left[\left(\frac{D_0}{Dt}\right)^2 - gH\nabla^2\right] h' = 0 \tag{10.24}$$

Similar to before, we make a guess that the solution is a plane wave such that $h' \sim \exp[i(kx + ly - \omega t)]$ and obtain the following dispersion relation:

$$(\omega - \mathbf{u}_0 \cdot \mathbf{k})^2 = (\omega - u_0 k - v_0 l)^2 = c_0^2 K^2, \quad \text{with} \quad K^2 = k^2 + l^2 \tag{10.25}$$

Thus, the effect of the constant background flow is to Doppler-shift the frequency by $-\mathbf{u_0} \cdot \mathbf{k}$.

10.3.2 Shallow water gravity waves: f-plane (Vallis 3.7.2)

We now extend the problem to include rotational effects with constant Coriolis parameter $f = f_0$, but keeping all of our other assumptions the same (e.g. no bottom topography). To keep things simple, let's once again assume a resting basic state: $u_0 = v_0 = 0$ so that $D_0/Dt = \partial_t$. (Remember that the background flow merely corresponds to adding a simple Doppler-shift to the wave frequency).

The linearized shallow water equations on the $f-$plane are then:

$$\partial_t u' - f_0 v' = -g\partial_x h', \qquad \partial_t v' + f_0 u' = -g\partial_y h', \qquad \partial_t h' + H(\partial_x u' + \partial_y v') = 0 \tag{10.26}$$

As before, we proceed by forming an equation for the divergence of the horizontal flow perturbations by adding the x-derivative of the u' equation to the y-derivative of the v' equation:

$$\partial_t u'_x - f_0 v'_x = -g\partial_{xx} h', \qquad \partial_t v'_y + f_0 u'_y = -g\partial_{yy} h', \tag{10.27}$$

$$\Rightarrow \quad \partial_t(u'_x + v'_y) + f_0(u'_y - v'_x) = -g\nabla_H^2 h' \tag{10.28}$$

Finally, we recall that the perturbation vorticity $\zeta' = v'_x - u'_y$ and so we have:

$$\partial_t(\nabla_H \cdot \mathbf{u}) - f_0\zeta' = -g\nabla_H^2 h' \tag{10.29}$$

From the continuity equation we have:

$$H\nabla_H \cdot \mathbf{u} = -\partial_t h' \tag{10.30}$$

and so solving for the divergence and plugging into the equation above we have

$$\partial_{tt} h' + f_0 H\zeta' = gH\nabla_H^2 h' \tag{10.31}$$

The goal is to have a wave equation for h' alone, and so, we need to get rid of the vorticity term. To do this, we go back to our original momentum equations and instead of making an equation for the divergence, we make an equation for the vorticity. This is done by taking the y-derivative of the u' equation and the

x-derivative of the v' equation and subtracting the first from the second. This yields:

$$\partial_t u'_y - f_0 v'_y = -g\partial_{xy} h', \qquad \partial_t v'_x + f_0 u'_x = -g\partial_{yx} h', \tag{10.32}$$

$$\Rightarrow \quad \partial_t (v'_x - u'_y) + f_0(u'_x + v'_y) = 0 \tag{10.33}$$

$$\Rightarrow \quad \partial_t \zeta' + f_0(\nabla_H \cdot \mathbf{u}) = 0 \quad \text{or} \quad f_0 H \partial_t \zeta' = f_0^2 \partial_t h' \tag{10.34}$$

where we have multiplied through by Hf_0 and used the fact that $H\nabla_H \cdot \mathbf{u} = -\partial_t h'$ to get to the final expression.

Putting everything together by taking the time derivative of 10.31 and using 10.34 to remove the dependence on ζ', we obtain a wave equation in h':

$$\partial_t \left[\partial_{tt} + f_0^2 - gH\nabla_H^2\right] h' = 0 \quad \text{or} \quad \partial_t \left[\partial_{tt} + f_0^2 - gH(\partial_{xx} + \partial_{yy})\right] h' = 0 \tag{10.35}$$

We make our usual plane wave assumption that $h' = \Re\{h_0 \exp[i(kx + ly - \omega t)]\}$ and we obtain the following dispersion relation:

$$\boxed{-i\omega[-\omega^2 + f_0^2 + gH(k^2 + l^2)] = 0} \tag{10.36}$$

The first solution, the trivial solution, is that $\omega = 0$. This situation means we have *steady flow*, and thus, going back to our shallow water momentum equations, that we are in geostrophic balance:

$$f_0 v' = g\partial_x h', \qquad f_0 u' = -g\partial_y h', \qquad \partial_x u' + \partial_y v' = 0 \tag{10.37}$$

The second solution is non-trivial, and can be found by setting the expression inside the brackets to zero. In this case, we obtain the dispersion relation for plane shallow water waves on the $f-$plane (so called *Poincaré waves*):

$$-\omega^2 + f_0^2 + gH(k^2 + l^2) = 0 \tag{10.38}$$

$$\Rightarrow \quad \omega^2 = f_0^2 + gH(k^2 + l^2) = f_0^2 + c_0^2 K^2 = f_0^2(1 + L_d^2 K^2) \tag{10.39}$$

where again $c_0 = \pm\sqrt{gH}$ (the phase speed of non-rotating shallow water gravity waves), $K^2 = k^2 + l^2$, and $L_d^2 \equiv c_0^2/f_0^2 = gH/f_0^2$.

L_d is called the *Rossby radius of deformation* and its meaning will become clear in the next section.

- **Large horizontal scales:** $L_d^2 K^2 << 1$, $\omega \to f_0$ but with $(\omega/f_0 > 1)$, i.e. inertial oscillations
- **Small horizontal scales:** $L_d^2 K^2 >> 1$, rotational effects become unimportant and $\omega \to c_0^2 K^2$ (as in the non-rotating case)

Note how the Rossby radius of deformation determines what appropriate "small" and "large" horizontal scales are.

10.3.3 small digression: Internal gravity waves

Gravity waves are ubiquitous in stably stratified geophysical fluids. As the name suggests, they arise due to the restoring force of gravity. The Boussinesq system (nearly incompressible - remember it from way back when?!) allows for a simple and insightful discussion of gravity waves within a fluid (rather than at the interface of the shallow water system). We make a mid-latitude f-plane assumption ($f = f_0$) to include rotational effects (gravity waves are of small enough scale in mid-latitudes so that variations of f with latitude are not important).

As we have done multiple times previously, we linearize our equations about a basic state in order to obtain wave solutions. Again, for simplicity, we will linearize about a state of rest so that $u = u', v = v', w = w'$, with constant background frequency $N^2 =$ constant. With these assumptions, further neglect all terms of higher-order in the perturbations. Note that is means that *all of the advection terms become quadratic and are thus removed by linearization.* Hence, the resulting linearized Boussinesq equations are:

$$\partial_t u' - f_0 v' = -\partial_x \phi' \tag{10.40}$$

$$\partial_t v' + f_0 u' = -\partial_y \phi' \tag{10.41}$$

$$\partial_t w' = -\partial_z \phi' + b' \tag{10.42}$$

$$\nabla \cdot \mathbf{v} = \partial_x u' + \partial_y v' + \partial_z w' = 0 \quad \textit{continuity equation} \tag{10.43}$$

$$\partial_t b' + N^2 w' = 0 \quad \textit{thermodynamic equation} \tag{10.44}$$

where

$$\phi' \equiv \frac{p'}{\rho_0}, \qquad b' \equiv -g\frac{\rho'}{\rho_0} \quad \text{and} \quad N^2 = -g\frac{1}{\rho_0}\frac{\partial \hat{\rho}(z)}{\partial z} \tag{10.45}$$

If you go back through Section 5.1 of these notes, you will find all of these equations hidden in the Boussinesq derivations.

These equations can be combined to form a single equation for the vertical velocity perturbation w':

$$\boxed{[\partial_{tt}(\partial_{xx} + \partial_{yy} + \partial_{zz}) + f_0^2 \partial_{zz} + N^2(\partial_{xx} + \partial_{yy})]w' = 0} \tag{10.46}$$

The procedure to derive this equation is to first combine the horizontal momentum equations to formulate equations for the divergence, $D' = \partial_x u' + \partial_y v'$, and vorticity, $\zeta' = \partial_x v' - \partial_y u'$, combining them to form an equation relating w' and ϕ', and then combining this latter equation with another such equation obtained from the vertical momentum and thermodynamic equations.

Note, the *hydrostatic approximation* gives a simplified vertical momentum equation ($b' = \partial_z \phi'$), which in turns gives a simplified thermodynamic equation ($\partial_{tz}\phi' + N^2 w' = 0$), which yields a simplified wave

equation:

$$\textit{hydrostatic approx.:} \quad (\partial_{tt} + f_0^2)\partial_{zz}w' + N^2(\partial_{xx} + \partial_{yy})w' = 0 \tag{10.47}$$

This represents a good example of how vertical fluctuations are not necessarily inconsistent with the hydrostatic approximation.

Special case: neglecting pressure perturbations. If we neglect the pressure perturbations (i.e. $\phi' = 0$), then we have from the vertical momentum equation that $\partial_t w' = b'$ and therefore

$$\partial_{tt}w' + N^2 w' = 0 \tag{10.48}$$

This equation is known as the *harmonic oscillator*, and describes vertical buoyancy oscillations with frequency N. You will explore the solution to the harmonic oscillator in your homework.

The general solution to 10.46 (non-hydrostatic, with pressure perturbations) can be found by seeking plane wave solutions, i.e.

$$w' = \Re[W \exp i(kx + ly + mz - \omega t)] \tag{10.49}$$

where k, l, m are the horizontal and vertical wave numbers and W is a constant amplitude.

Following similar steps to our other problems, we obtain the following *gravity wave dispersion relation:*

$$\omega^2 = \frac{N^2K^2 + f_0^2 m^2}{K^2 + m^2}, \quad \text{where} \quad K^2 = k^2 + l^2 \tag{10.50}$$

Note that since we have assumed a resting basic state, this is the equation for the intrinsic wave frequency. A constant background flow would simply Doppler shift this.

For Earth in midlatitudes, N tends to be at least one order of magnitude larger than f_0 (typically $N/f_0 \sim 100$). Thus, we can distinguish two cases:

Rotation is important: For rotation to be important the vertical wavelength ($\sim 1/m$) needs to be one or two orders of magnitude smaller than the total horizontal wavelength ($\sim 1/K$). Gravity waves for which f_0 is important are often called *interia gravity waves*.

Rotation is not important: Most waves, however, are isotropic, meaning that their vertical and horizontal wavelengths are of similar magnitudes. Thus, in most instances rotation can be neglected.

Without rotation ($f_0 = 0$), it is sufficient to consider two spatial dimensions (i.e. x and z and set $l = 0$). Then the dispersion relation reduces to:

$$\textit{2D non-rotating:} \quad \omega^2 = \frac{N^2k^2}{k^2 + m^2} = N^2\frac{k^2}{\kappa^2} \quad \text{where} \quad \kappa^2 = k^2 + m^2 \tag{10.51}$$

For the phase speed components of these non-rotating Boussinesq gravity waves we obtain:

$$\textit{2D non-rotating:} \quad c_p^x = \pm\frac{N}{\kappa}, \qquad c_p^z = \frac{\omega}{m} = \frac{k}{m}c_p^x = \pm\frac{k}{m}\frac{N}{\kappa} \tag{10.52}$$

and for the group velocities:

$$\textit{2D non-rotating:} \quad c_g^x = \pm\frac{m^2}{\kappa^2}\frac{N}{\kappa} = \frac{m^2}{\kappa^2}c_p^x \tag{10.53}$$

$$c_g^z = \pm\frac{-km}{\kappa^2}\frac{N}{\kappa} = \frac{-km}{\kappa^2}c_p^x \tag{10.54}$$

- the vertical phase speed and vertical group velocity have opposite sign: for phase lines propagating downward, energy propagates upward!
- the group velocity points along lines of constant phase (perpendicular to the phase speed)! i.e.,

$$(k, m) \cdot \mathbf{c_g} = kc_g^x + mc_g^z = 0. \tag{10.55}$$

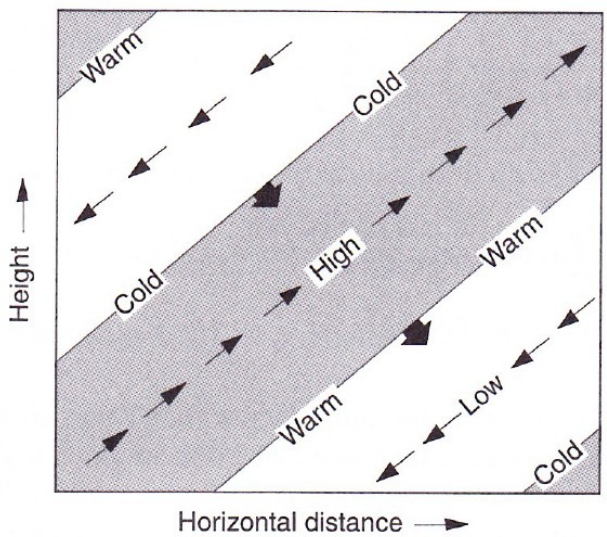

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Fig. 7.9 Idealized cross section showing phases of pressure, temperature, and velocity perturbations for an internal gravity wave. Thin arrows indicate the perturbation velocity field, blunt solid arrows the phase velocity. Shading shows regions of upward motion.

Finally, we expect the non-rotating, two-dimensional version of the dispersion relation to apply well to mesoscale gravity waves. Thus, in the limit of horizontal scales much smaller than vertical scales ($k^2 >> m^2$), we obtain:

$$\omega^2 \approx N^2 \tag{10.56}$$

which corresponds to the *high-frequency* limit for waves which corresponds to the simple buoyancy oscillations (harmonic oscillator of 10.48) described above.