10.6 Geostrophic adjustment (Vallis 3.8, Schubert Ch. 10)

Why is geostrophic balance so dominant in the large-scale motions of the atmosphere and oceans? Why is the Rossby number so small? To approach this question, we will consider a situation of an initial flow that *is not in geostrophic balance* and we will study what happens to this flow over time. We will show that the flow will *adjust to a state of geostrophic balance*, a process known as *geostrophic adjustment*. This adjustment process occurs generally in rotating fluids, whether they are compressible or not, stratified or not.

The shallow water equations provide a simple system for studying geostrophic adjustment using potential vorticity (PV) conservation. Suppose the fluid is initially at rest but has a sharp discontinuity in surface height:

$$h'(\mathbf{x}, t=0) = -\eta_0 \,\mathrm{sgn}\,(x) = \eta_0 \begin{cases} +1 & \text{if} \quad x < 0 \\ -1 & \text{if} \quad x > 0 \end{cases} \tag{10.79}$$

We now ask, how will the surface height h' evolve in time and what is its final equilibrium structure? We will show that the answer to this depends fundamentally on whether we include rotation or not. We will begin with the non-rotating case even though it will clearly not tell us anything about geostrophic balance.

show youtube video of adjustment: https://www.youtube.com/watch?v=FkuaTnVOBPA

10.6.1 Non-rotating Case

We begin with our assumptions. First, since all horizontal directions are symmetric in the absence of rotation, we will only work in the x-direction without loss of generality. Next, we assume that $u(x, t = 0) = 0$ everywhere and that the linearized shallow water equations apply (i.e. the surface displacements are small).

In the non-rotating case we expect non-dispersive gravity waves to be formed from our initial perturbation. Recall that these gravity waves travel with speed $c = c_0 = \pm\sqrt{gH}$. In the case of non-dispersive waves, the waves propagate as though they are just being advected. Thus, the height displacements of the fluid h' at any given time can be written generally as

$$h'(x,t) = \frac{1}{2}\left(F(x + c_0 t) + F(x - c_0 t)\right) \tag{10.80}$$

where

$$F(x) = h'(x, t=0) = -\eta_0 \,\mathrm{sgn}\,(x)\,. \tag{10.81}$$

Thus, h' is given by

$$\boxed{h'(x,t) = -\frac{\eta_0}{2}\left(\text{sgn}\,(x + c_0 t) + \text{sgn}\,(x - c_0 t)\right)} \tag{10.82}$$

and describes height perturbations that split into two fronts that propagated away to the left and right.

See Matlab example of perturbation split

The velocity field at location x and time t can be obtained from the h' equation using the shallow water momentum equation

$$\partial_t u' = -g\partial_x h' \tag{10.83}$$

Using our equation for the evolution of h', we obtain the evolution equation for u' (if you don't believe me, test it and make sure the momentum equation above will be obtained):

$$\boxed{u'(x,t) = \frac{g\eta_0}{2c_0}\left(\text{sgn}\,(x + c_0 t) - \text{sgn}\,(x - c_0 t)\right)} \tag{10.84}$$

The above solution represents situation where the initial perturbation is translated to the right and left at speed c_0. Thus, at any given location away from the initial disturbance, the fluid will remain at rest until the front arrives. After the front has passed, it will return to its resting state. That is, the disturbance is radiated away completely.

Show Vallis Fig. 3.9

10.6.2 Rotating Case

Now consider the rotating case, that is, flow on an f-plane. In this instance, potential vorticity conservation tells us that

$$\frac{Dq}{Dt} = 0 \quad \text{with} \quad q = \frac{\zeta + f_0}{h} = \frac{\zeta + f_0}{H + h'} \tag{10.85}$$

and it turns out that this will provide a powerful constraint on the evolution of the flow.

The linearized PV on an f-plane is given by 10.70:

$$q \approx \frac{f_0}{H} + \frac{\tilde{q}}{H} \quad \text{where} \quad \tilde{q} \equiv \zeta' - f_0\frac{h'}{H} = \left(\nabla^2 - \frac{1}{L_d^2}\right)\psi' \tag{10.86}$$

with the streamfunction $\psi' = gh'/f_0$ and $L_d^2 = (c_0/f_0)^2$.

Since f_0 is constant and there is no initial basic flow (as for the non-rotating case), PV conservation simply reduces to:

$$\frac{\partial \tilde{q}}{\partial t} = 0. \tag{10.87}$$

This is an incredibly strong statement about PV - it tells us that the PV will be unchanged at every location in the fluid during the entire adjustment process! Thus, since our initial $\tilde{q}$ field is given by

$$\tilde{q}(x, t = 0) = \frac{f_0 \eta_0}{H} \operatorname{sgn}(x), \tag{10.88}$$

it will remain this value for all t.

PV conservation in the non-rotating case

You may be thinking, "hold on!", PV is also conserved in the non-rotating case. Thus, this "PV fixed in space result" was in-play there too. This is true. However, in the non-rotating case $q = \zeta = 0$ and thus the "fixed PV" constraint merely said that our initially irrotational fluid would remain irrotational. We will see that this constraint is far stronger in the rotating case.

Returning to 10.88, we can set the PV at the initial time equal to the PV at a later time and obtain:

$$(\partial_{xx} - \frac{1}{L_d^2})\psi' = \frac{f_0 \eta_0}{H} \operatorname{sgn}(x) \tag{10.89}$$

Note that we have dropped the derivatives w.r.t. y since the flow will remain uniform in y (no forces causing changes in that direction).

The above equation is part of a set known as *inhomogeneous equations*; i.e. its solution can be obtained as the general solution to its homogenous part plus a particular solution. We first solve the homogenous part of this equation, i.e.

$$(\partial_{xx} - \frac{1}{L_d^2})\psi' = 0 \tag{10.90}$$

and obtain the solution

$$\psi'_h = Ae^{x/L_d} + Be^{-x/L_d} \tag{10.91}$$

The inhomogeneity (i.e. the right-hand-side of our differential equation) is piecewise constant and will therefore result in a similarly piecewise additive constant to our solution, namely, $-g\eta_0/f_0 \operatorname{sgn}(x)$.

Now we need to determine our constants by first constraining them separately for $x < 0$ and $x > 0$, by requiring that the solution remain finite as $|x| \to \infty$, and then setting these two solutions equal to one another at $x = 0$ (i.e. continuous boundary conditions). That is, $B = 0$ for $x < 0$ and $A = 0$ for $x > 0$. At $x = 0$, $\psi' = 0$ and we set the derivatives of each side equal to each other $[\partial_x \psi'|_{x=0} = -g\eta_0/(L_d f_0)]$ to get a constant factor $g\eta_0/f_0 \operatorname{sgn}(x)$ out in front. Thus the full solution for the final, geostrophically adjusted state is:

$$\boxed{\psi' = -\frac{g\eta_0}{f_0} \operatorname{sgn}(x) \left(1 - e^{-|x|/L_d}\right)} \tag{10.92}$$

or

$$\boxed{\psi' = \begin{cases} -\frac{g\eta_0}{f_0}\left(1 - e^{-x/L_d}\right) & x < 0 \\ +\frac{g\eta_0}{f_0}\left(1 - e^{x/L_d}\right) & x > 0 \end{cases}} \tag{10.93}$$

The velocity field in this balanced state is obtained by the definition of the streamfunction. That is,

$$\boxed{u' = -\partial_y \psi' = 0 \quad \text{and} \quad v' = \partial_x \psi' = -\frac{g\eta_0}{f_0 L_d} e^{-|x|/L_d}} \tag{10.94}$$

This steady state solution describes a sloped surface with a jet along the initial discontinuity. The slope is maintained by a balance between gravity trying to flatten the slope and angular momentum conservation trying to keep fluid parcels close to their initial position. The deformations of surface height and the fluid flow *only extend out to* L_d - this is where the name "deformation radius" comes from. Another way to say this is that the variations in height are not radiated away to infinity as in the non-rotating case. Note further that even though these are steady state solutions, they cannot be obtained by simply setting the time derivatives to zero in the original equations (which would merely give geostrophic balance of *any* streamfunction). It is the combination of geostrophic balance together with PV conservation that gives us one specific streamfunction solution.

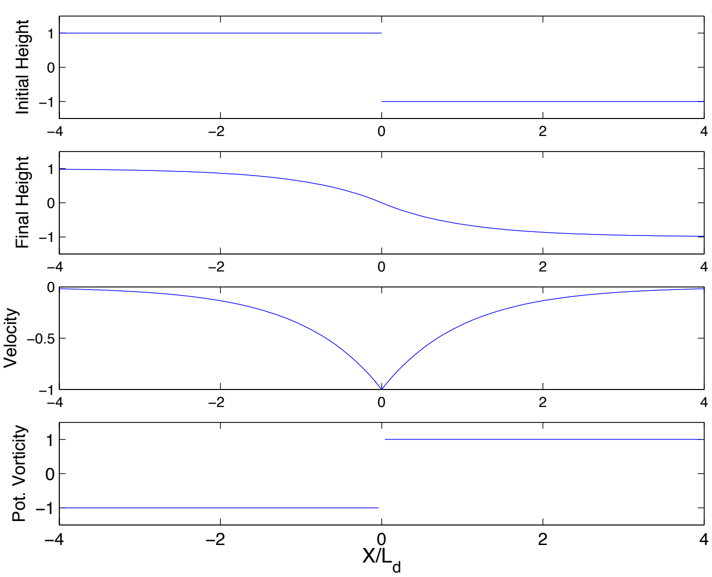

Fig. 3.8 Solutions of a linear geostrophic adjustment problem. Top panel: the initial height field, given by (3.125) with $\eta_0 = 1$. Second panel: equilibrium height field, η given by (3.137) and $\eta = f_0\psi/g$. Third panel: equilibrium geostrophic velocity (normal to the gradient of height field), given by (3.138). Bottom panel: potential vorticity, given by (3.133), and this does note evolve. The distance, x is non-dimensionalized by the deformation radius $L_d = \sqrt{gH}/f_0$, and the velocity by $\eta_0(g/f_0L_d)$. Changes to the initial state occur only within $\mathcal{O}(L_d)$ of the initial discontinuity; and as $x \to \pm\infty$ the initial state is unaltered.

10.7 Energetics of geostrophic adjustment (Vallis 3.8, Schubert Ch. 10)

10.7.1 General considerations

Since density is constant in the shallow water system ($\rho = \rho_0$), no thermodynamic equation is needed (i.e. no internal energy changes). Thus, we expect the energetics of the system to only involve potential and kinetic energy. For the following derivation we will assume no bottom topography, and thus, $\nabla\eta = \nabla h$.

The potential energy *per unit volume* is given by $\rho_0 gz$, which upon vertical integration gives the potential energy *per unit area* (denoted from now on as "PE"):

$$\boxed{\mathrm{PE} = \int_0^h \rho_0 gz\, dz = \frac{1}{2}\rho_0 g\int_0^h (dz)^2 = \frac{1}{2}\rho_0 g h^2} \tag{10.95}$$

In order to obtain an equation for PE we multiply the shallow water continuity equation by $\rho_0 gh$:

$$\rho_0 gh[\partial_t h + \nabla\cdot(h\mathbf{u})] = 0 \qquad \Rightarrow \qquad \partial_t \mathrm{PE} + \nabla\cdot(\mathbf{u}\mathrm{PE}) + \mathrm{PE}\nabla\cdot\mathbf{u} = 0 \tag{10.96}$$

The kinetic energy *per unit volume* is given by $\rho_0\mathbf{u}/2$, which upon vertical integration gives the kinetic energy *per unit area* (denoted from now on as "KE"):

$$\mathrm{KE} = \int_0^h \frac{1}{2}\rho_0\mathbf{u}^2\, dz = \frac{1}{2}\rho_0 h\mathbf{u}^2 \tag{10.97}$$

where we have used the fact that $\mathbf{u}$ is independent of z. In order to obtain an evolution equation for KE we multiply the shallow water momentum equations by $\rho_0 h\mathbf{u}$:

$$\begin{aligned} \rho_0 h\mathbf{u}[\partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \mathbf{f}\times\mathbf{u} &= -g\nabla h] \qquad \Rightarrow && (10.98)\\ \partial_t\mathrm{KE} - \frac{1}{2}\rho_0\mathbf{u}\partial_t h + (\mathbf{u}\cdot\nabla)\mathrm{KE} - \frac{1}{2}\rho_0\mathbf{u}^2(\mathbf{u}\cdot\nabla)h &= -g\rho_0(\mathbf{u}\cdot\nabla)\frac{h^2}{2} = -(\mathbf{u}\cdot\nabla)\mathrm{PE} && (10.99) \end{aligned}$$

Recall that mass continuity is written as $Dh/Dt = -h\nabla\cdot\mathbf{u}$ and so, we can simplify our equation for $\partial_t\mathrm{KE}$ to be:

$$\boxed{\partial_t\mathrm{KE} + \nabla\cdot(\mathbf{u}\mathrm{KE}) + (\mathbf{u}\cdot\nabla)\mathrm{PE} = 0.} \tag{10.100}$$

Adding our evolution equations for PE and KE gives an equation for the total energy per unit area (denoted as "E"):

$$\partial_t E + \nabla\cdot[\mathbf{u}(E + \mathrm{PE})] = \partial_t E + \nabla\cdot\mathbf{F} = 0 \tag{10.101}$$

where $E = \mathrm{PE} + \mathrm{KE}$ and $\mathbf{F} = \mathbf{u}(E + \mathrm{PE})$ which is the *energy flux per unit area.* Note that the energy flux is not simply velocity times energy (as one may naively think). The extra PE term corresponds to the fluid doing work against the pressure gradient force (in shallow water $p = \rho_0 gz$, i.e. the pressure equals the potential energy per unit volume). Integrating the energy over the size of the domain we must have that total energy is conserved (with vanishing velocity normal to the boundaries, or periodic boundary conditions, such that the integral over the divergence term vanishes):

$$\frac{\partial \hat{E}}{\partial t} = 0, \quad \text{with} \quad \hat{E} \equiv \int\int E \, dx dy \tag{10.102}$$

10.7.2 Energetics for Geostrophic Adjustment

The linearized shallow water equations with zero basic state flow are:

$$\partial_t \mathbf{u}' + \mathbf{f} \times \mathbf{u}' = -g\nabla h', \qquad \partial_t h' + H\nabla \cdot \mathbf{u}' = 0 \tag{10.103}$$

Applying the same operations as before gives the energy equation:

$$\partial_t \left(\frac{1}{2}\rho_0 H \mathbf{u}'^2 + \frac{1}{2}\rho_0 g h'^2 \right) + \rho_0 g H \nabla \cdot (\mathbf{u} h') = 0 \tag{10.104}$$

The initial (available) potential energy per unit area is:

$$\int_{-\infty}^{\infty} \frac{1}{2} g\eta_0^2 \, dx = \int_{0}^{\infty} g\eta_0^2 \, dx = \infty \tag{10.105}$$

where we have used symmetry about $x = 0$ (and will do so from now on for all other horizontal integrals).

In the non-rotating case the initial PE is completely converted into KE (the gravity shockwave propagating away). However, in the rotating case some of the initial PE *remains* as PE due to the PV conservation/angular momentum constraint. The total energy at the initial time is equal to the total PE (no KE), and we have already discussed that this in infinite. However, the difference between the final and initial PE is finite and is:

$$\mathrm{PE}_f - \mathrm{PE}_i = \rho_0 g\eta_0^2 \int_0^{\infty} [(1 - e^{-x/L_d})^2 - 1] \, dx = -\frac{3}{2} g\eta_0^2 \rho_0 L_d \tag{10.106}$$

where we have used the equation for our final streamfunction and the fact that $\psi = g\eta/f \Rightarrow \eta = \psi f/g$.

The KE density of the final state is:

$$\mathrm{KE}_f - \mathrm{KE}_i = H\rho_0 \int_0^{\infty} \mathbf{u}^2 \, dx - 0 = \frac{1}{2} g\eta_0^2 \rho_0 L_d, \tag{10.107}$$

where we have used our knowledge of the velocity field at the final time 10.94 and that $L_d^2 = gH/f^2$.

That is, only 1/3 of the total released PE is converted into KE of the geostrophic final state, with the remainder ($g\eta_0^2\rho_0 L_d$) being radiated away to infinity as gravity waves.