to express the equations of motion, so far as is possible, in so-called 'non-dimensional' variables, by which we mean expressing every variable (such as velocity) as the ratio of its value to some reference value. For velocity the reference could, for example, be the speed of of light — but this would not be very helpful for fluid dynamical problems in the Earth's atmosphere or oceans! Rather, we should choose the reference value as a natural one for a given flow, in order that, so far as possible, the non-dimensional variables are order-unity quantities, and doing this is called *scaling the equations*. Evidently, there is no reference velocity that is universally appropriate, and much of the art of fluid dynamics lies in choosing sensible scaling factors for the problem at hand. Non-dimensionalization plays an important role in fluid dynamics, and we introduce it here with a simple example.

1.11.1 The Reynolds number

Consider the constant-density momentum equation in Cartesian coordinates. If a typical velocity is U, a typical length is L, a typical time scale is T, and a typical value of the pressure deviation is Φ, then the approximate sizes of the various terms in the momentum equation are given by

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\nabla\phi + \nu\nabla^2\boldsymbol{v} \tag{1.194a}$$

$$\frac{U}{T} \qquad \frac{U^2}{L} \quad \sim \quad \frac{\Phi}{L} \qquad \nu\frac{U}{L^2}. \tag{1.194b}$$

The ratio of the inertial terms to the viscous terms is $(U^2/L)/(\nu U/L^2) = UL/\nu$, and this is the *Reynolds number*.[12] More formally, we can non-dimensionalize the momentum equation by writing

$$\hat{\boldsymbol{v}} = \frac{\boldsymbol{v}}{U}, \qquad \hat{\boldsymbol{x}} = \frac{\boldsymbol{x}}{L}, \qquad \hat{t} = \frac{t}{T}, \qquad \hat{\phi} = \frac{\phi}{\Phi}, \tag{1.195}$$

where the terms with hats on are *non-dimensional* values of the variables and the capitalized quantities are known as *scaling values*, and these are the approximate magnitudes of the variables. We choose the non-dimensionalization so that the non-dimensional variables are of order unity. Thus, for example, we choose U so that $u = \mathcal{O}(U)$, where this should be taken to mean that the magnitude of the variable u is approximately U, or that $u \sim U$, and we say that 'u scales like U'. [This $\mathcal{O}()$ notation differs from the conventional mathematical meaning of 'order', in which $a = O(\epsilon^\alpha)$ represents a limit in which $a/\epsilon^\alpha \to$ constant as $\epsilon \to 0$.] Thus, if there are well-defined length and velocity scales in the problem, the non-dimensional variables are of order unity; that is, $\hat{u} = \mathcal{O}(1)$, and similarly for the other variables.

Because there are no external forces in this problem, appropriate scaling values for time and pressure are

$$T = \frac{L}{U}, \qquad \Phi = U^2. \tag{1.196}$$

Substituting (1.195) and (1.196) into the momentum equation we obtain

$$\frac{U^2}{L}\left[\frac{\partial \hat{\boldsymbol{v}}}{\partial \hat{t}} + (\hat{\boldsymbol{v}}\cdot\nabla)\hat{\boldsymbol{v}}\right] = -\frac{U^2}{L}\nabla\hat{\phi} + \frac{\nu U}{L^2}\nabla^2\hat{\boldsymbol{v}}, \tag{1.197}$$

where we use the convention that when ∇ operates on a non-dimensional variable it is a non-dimensional operator. Equation (1.197) simplifies to

$$\frac{\partial \hat{\boldsymbol{v}}}{\partial \hat{t}} + (\hat{\boldsymbol{v}}\cdot\nabla)\hat{\boldsymbol{v}} = -\nabla\hat{\phi} + \frac{1}{Re}\nabla^2\hat{\boldsymbol{v}}, \tag{1.198}$$

where

$$Re \equiv \frac{UL}{\nu} \tag{1.199}$$

is, again, the Reynolds number. If we have chosen our length and velocity scales sensibly — that is, if we have scaled them properly — each variable in (1.198) is order unity, with the viscous term being multiplied by the parameter $1/Re$. There are two important conclusions:

(i) The ratio of the importance of the inertial terms to the viscous terms is given by the *Reynolds number*, defined by (1.199). In the absence of other forces, such as those due to gravity and rotation, the Reynolds number is the only non-dimensional parameter explicitly appearing in the momentum equation. Hence its value, along with the boundary conditions, controls the behaviour of the system.

(ii) More generally, by scaling the equations of motion appropriately the parameters determining the behaviour of the system become explicit. *Scaling the equations is intelligent non-dimensionalization.*

Notes

1 Joseph-Louis Lagrange (1736–1813) was a Franco-Italian, born and raised in Turin who then lived and worked mainly in Germany and France. He made notable contributions in analysis, number theory and mechanics and was recognized as one of the greatest mathematicians of the eighteenth century. He laid the foundations of the calculus of variations (to wit, the 'Lagrange multiplier') and first formulated the principle of least action, and his treatise *Mécanique Analytique* (1788) provides a unified analytic framework (it contains no diagrams, a feature emulated in Whittaker's *Treatise on Analytical Dynamics*, 1927) for all Newtonian mechanics.

Leonard Euler (1707–1783), a Swiss mathematician who lived and worked for extended periods in Berlin and St. Petersburg, made important contributions in many areas of mathematics and mechanics, including the analytical treatment of algebra, the theory of equations, calculus, number theory and classical mechanics. He was the first to establish the form of the equations of motion of fluid mechanics, writing down both the field description of fluids *and* what we now call the material or advective derivative.

Truesdell (1954) points out that 'Eulerian' and 'Lagrangian', especially the latter, are inappropriate eponyms. The so-called Eulerian description was introduced by d'Alembert in 1749 and generalized by Euler in 1752, and the so-called Lagrangian description was introduced by Euler in 1759. The modern confusion evidently stems from a monograph by Dirichlet in 1860 that credits Euler in 1757 and Lagrange in 1788 for the respective methods. Clifford Truesdell (1919–2000) was a remarkable figure himself, known both for his own contributions to many areas of continuum mechanics and for his scholarly investigations on the history of mathematics and science; he also had a trenchant and at times pungent writing style. See Ball & James (2002) for more details.