DERIVATION OF VORTICITY TENDENCY EQUATION

Vorticity: $\vec{\omega} = \nabla \times \vec{\omega}$

$$\vec{\omega} = (\partial_y w - \partial_z v)\hat{\imath} - (\partial_x w - \partial_z u)\hat{\jmath} + (\partial_x v - \partial_y u)\hat{k}$$

Vertical component: $\zeta \equiv (\partial_x v - \partial_y u)\hat{k}$

Want a time tendency (prognostic) eq. for ζ.

- ζ is a spatial derivative of $\vec{v}$
- Momentum equations are time tendency of $\vec{v}$
- time tendency of ζ → we need to take a spatial derivative of the $\vec{v}$ tendency equations (horizontal momentum)!

<u>Step 1</u>: rewrite horizontal advection terms:

$$u: \quad u\partial_x u + v\partial_y u = -v\zeta + \tfrac{1}{2}\left[\partial_x(u^2+v^2)\right]$$

$$= -v(\partial_x v - \partial_y u) + \tfrac{1}{2}\left[\partial_x(u^2+v^2)\right]$$

u, v depend on x, y; so

$$u\partial_x u + v\partial_y u = -v(\partial_x v - \partial_y u) + \tfrac{1}{2}(2u\partial_x u + 2v\partial_x v)$$

$$= -v(\cancel{\partial_x v} - \partial_y u) + u\partial_x u + \cancel{v\partial_x v}$$

$$u\partial_x u + v\partial_y u = +v\partial_y u + u\partial_x u$$

similarly,

$$v: \quad u\partial_x v + v\partial_y v = u\zeta + \tfrac{1}{2}\left[\partial_y(u^2+v^2)\right]$$

Now, we have horizontal advection in terms of ζ!

2. Substitute results from 1. into u & v mom^{m} eqs.

$$(1)\ u: \partial_t u - v\zeta - vf + w\partial_z u = -\frac{1}{\rho}\partial_x P - \frac{1}{2}\partial_x(u^2+v^2) + X$$

$$(2)\ v: \partial_t v + u\zeta + uf + w\partial_z v = -\frac{1}{\rho}\partial_y P - \frac{1}{2}\partial_y(u^2+v^2) + Y$$

conservative forces (i.e. friction, viscous forces)

3. Get ζ tendency equation

- ζ is horizontal derivative of $\vec{v}$
- momentum equations are $\vec{v}$ tendency
- so, for ζ tendency, we want horiz. derivatives of momentum equations:

$$\partial_x (2) - \partial_y (1)$$

$$\partial_x (2): \partial_{x,t} v + \partial_x[u(\zeta+f)] + \partial_x(w\partial_z v) = \partial_x\left[-\frac{1}{\rho}\partial_y P\right] - \frac{1}{2}\partial_{x,y}(u^2+v^2) + \partial_x Y$$

LHS: $\partial_{x,t} v + (\zeta+f)\partial_x u + u\partial_x(\zeta+f) + w\partial_{x,z} v + \partial_x w \partial_z v$

RHS: $-\frac{1}{\rho}\partial_{x,y}P + \partial_y P \partial_x\left(-\frac{1}{\rho}\right) - \frac{1}{2}\left[\partial_{x,y}(u^2+v^2)\right] + \partial_x Y$

$$\partial_y (1): \partial_{y,t} u - \partial_y[v(\zeta+f)] + \partial_y(w\partial_z u) = \partial_y\left[-\frac{1}{\rho}\partial_x P\right] - \frac{1}{2}\partial_{y,x}(u^2+v^2) + \partial_y X$$

LHS: $\partial_{y,t} u - (\zeta+f)\partial_y v - v\partial_y(\zeta+f) + w\partial_{y,z} u + \partial_y w \partial_z u$

RHS: $-\frac{1}{\rho}\partial_{y,x}P + \partial_x P \partial_y\left(-\frac{1}{\rho}\right) - \frac{1}{2}\left[\partial_{y,x}(u^2+v^2)\right] + \partial_y X$

$\partial_x (2) - \partial_y (1)$, combine LHS & RHS

LHS: $\partial_{x,t} v - \partial_{y,t} u + (\zeta+f)(\partial_x u + \partial_y v) + u\partial_x(\zeta+f) + v\partial_y(\zeta+f)$

$+ w(\partial_{x,z} v - \partial_{y,z} u) + \partial_x w \partial_z v - \partial_y w \partial_z u$

Recall, we can rewrite derivatives:

$$\partial_{x,t} v \Longleftrightarrow \partial_{t,x} v \quad \left(\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t}\right) \Longleftrightarrow \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x}\right)\right)$$

So, we can rewrite the LHS further:

$$\text{LHS}: \partial_t(\partial_x v - \partial_y u) + u\partial_x(\zeta+f) + v\partial_y(\zeta+f) + w[\partial_z(\partial_x v - \partial_y u)] + \ldots$$
$$+(\zeta+f)(\partial_x u + \partial_y v) + \partial_x w \partial_z v - \partial_y w \partial_z u$$

Now, we can substitute ζ in, and rewrite:

$$\partial_t \zeta + u\partial_x(\zeta+f) + v\partial_y(\zeta+f) + w\partial_z \zeta + (\zeta+f)(\partial_x u + \partial_y v) + \ldots$$
$$+\partial_x w \partial_z v - \partial_y w \partial_z u$$

$$\frac{D}{Dt} \equiv \partial_t + u\partial_x + v\partial_y + w\partial_z, \text{ so:}$$

$$\frac{D}{Dt}(\zeta+f) + (\zeta+f)(\partial_x u + \partial_y v) + \partial_x w \partial_z v - \partial_y w \partial_z u : \text{LHS}$$

since f does not depend on t or z, we can put it inside $\frac{D}{Dt}$

Now, we look at the RHS:

$$\partial_y(\rho^{-1}) = \frac{1}{\rho^2}\partial_y \rho$$

$$\text{RHS}: \frac{-1}{\rho}(\cancel{\partial_{x,y}P - \partial_{y,x}P}) + \partial_y P \partial_x\left(\frac{-1}{\rho}\right) - \partial_x P \partial_y\left(\frac{-1}{\rho}\right) - \frac{1}{2}(\cancel{\partial_{x,y}(u^2+w^2) - \partial_{y,x}(u^2+w^2)})$$
$$0 \qquad\qquad 0$$
$$+\partial_x Y - \partial_y X$$

4. Combine LHS & RHS

$$\frac{D}{Dt}(\zeta+f) + (\zeta+f)(\partial_x u + \partial_y v) + \partial_x w \partial_z v - \partial_y w \partial_z u = \partial_y P \partial_x\left(\frac{-1}{\rho}\right) - \partial_x P \partial_y\left(\frac{-1}{\rho}\right) + \partial_x Y - \partial_y X$$

rearrange one more time:

$$\frac{D}{Dt}(\zeta+f) = \underset{①}{-(\zeta+f)(\partial_x u + \partial_y v)} + \underset{②}{[\partial_z u \partial_y w - \partial_z v \partial_x w]} + \underset{③}{\left[\frac{1}{\rho^2}(\partial_y P \partial_x \rho - \partial_x P \partial_y \rho)\right]} + \underset{④}{(\partial_x Y - \partial_y X)}$$

1) Stretching / horizontal divergence
2) tilting (by vertical shear of u, v or horizontal shear of w)
3) solenoidal / baroclinic term (vertical component of $\frac{(\nabla\rho \times \nabla P)}{\rho^2}$)
4) nonconservative terms (e.g. friction, viscosity)

Conservation of absolute vorticity

$\frac{D}{Dt}(\zeta + f) = 0$ when:

- incompressible fluid (1 is 0)
- conservative (4 is 0)
- barotropic (3 is 0)
- 2-dimensional (2 is 0)