Quantifying Isentropic Mixing in a Modified Lagrangian Coordinate:

Applications to Rossby Wave Breaking

Chengji Liu* and Elizabeth A. Barnes

Department of Atmospheric Sciences, Colorado State University, Fort Collins, Colorado

*Corresponding author address: Department of Atmospheric Sciences, Colorado State University, Fort Collins, Colorado.

E-mail: cjliu@atmos.colostate.edu
ABSTRACT

Isentropic mixing is an important process for the distribution of chemical constituents in the mid-to-high latitudes. A new framework is introduced to quantify both symmetric and asymmetric isentropic mixing in a modified Lagrangian coordinate. The framework is applied to quantify the mixing associated with the two types of Rossby wave breaking (i.e. cyclonic and anticyclonic) in two sets of idealized numerical simulations – eddy lifecycle simulations and a long climate simulation. In both sets of simulations, cyclonic wave breaking (CWB) exhibits either comparable or stronger mixing than anticyclonic wave breaking (AWB). Since CWB occurs less frequently while AWB occurs more frequently when the jet shifts poleward, this difference in mixing strength between the two types of Rossby wave breaking translates into a change in total isentropic mixing when the jet shifts. Specifically, it is found in the climate simulation that as the jet shifts poleward, symmetric mixing is weakened in the underworld and both symmetric and asymmetric mixing across the tropopause is weakened in the mid-to-high latitudes. The dynamical relationship above is relevant not only for internal climate variability such as the El-Niño Southern Oscillation (ENSO) and the annular modes, but also, for future climate change that may drive changes in the jet position.
1. Introduction

Atmospheric transport and mixing play a fundamental role in the global distribution of chemical pollutants and moisture. In the extratropics, transport and mixing by eddies are known to mainly operate along isentropic surfaces, i.e. constant potential temperature ($\theta$) surfaces. Within the troposphere, warm conveyor belts embedded in extratropical cyclones transport warm moist air poleward and upward along sloped isentropes (e.g. Thornicroft et al., 1993; Madonna et al., 2014). These transport events are responsible for poleward moisture transport (e.g. Eckhardt et al., 2004) and pollutant transport into the Arctic (e.g. Raatz and Shaw, 1984; Barrie, 1986). In the upper troposphere, vigorous exchange between stratospheric and tropospheric constituents happens along isentropes across a folding tropopause (e.g. Shapiro, 1980). Such exchange is an important source of ozone in the troposphere (e.g. Lelieveld and Dentener, 2000) where it is considered a pollutant and is detrimental to human health (e.g. Lippmann, 1989). This exchange also injects anthropogenic trace species such as chlorofluorocarbon (CFC) into the stratosphere (Holton et al., 1995).

Due to the dominance of isentropic transport and mixing in the extratropics, $\theta$ is often used as the vertical coordinate in quantifying these processes. Under the adiabatic approximation, $\theta$ can be considered a material surface and hence a Lagrangian coordinate. A Lagrangian coordinate is a more natural choice than a Eulerian one for transport and mixing since it avoids the need for Stokes correction for artifacts arising from the Eulerian-mean (McIntyre, 1980). With $\theta$ being the vertical coordinate, reversible vertical oscillations are absent and any vertical flux is associated with diabatic heating. Along these same lines, it is natural to also choose a conserved variable as the meridional coordinate so that any meridional flux in such a coordinate is only associated with non-conservative processes, such as horizontal diffusion. Examples include using potential vor-
ticity (PV) or the concentration of a passive tracer as the meridional coordinate (e.g. Butchart and
Remsberg, 1986). By integrating quantities between contours of a conserved quantity, a general
description of the irreversible mixing is possible without expensive trajectory calculations.

In such a Lagrangian coordinate, diffusive processes can alter the budget of a quantity within a
coordinate contour in two ways. The first way is by changing the mass enclosed by a contour. As
we will show later, this is done by net mass flux across the contour through what we call asymmet-
ric mixing. The other way is by changing the concentration of the quantity within a contour. This
is done by down-gradient mixing across the contour which we call symmetric mixing. However,
we are not aware of any study that has quantified both processes in a modified Lagrangian coor-
dinate. Nakamura (1995) proposed the modified Lagrangian mean (MLM) continuity equation to
quantify net mass flux due to non-conservative processes, which specifically isolates asymmetric
mixing in the horizontal direction. More recent studies that use this framework (e.g. Nakamura,
2004, 2007; ?) are also focused sorely on asymmetric mixing. On the other hand, Nakamura
(1996) studied the distribution of a passive tracer in non-divergent barotropic flow and used the
mass (i.e. area) enclosed by the tracer as the meridional coordinate. In such a coordinate, the net
mass flux is zero by construction and symmetric mixing alone determines the distribution of the
passive tracer. Under these conditions, he derived the MLM tracer equation in a diffusive form and
proposed the “effective diffusivity”. Other studies that have used the effective diffusivity (or its
generalized form) also focus on symmetric mixing (e.g. Haynes and Shuckburgh, 2000a,b; Chen
and Plumb, 2014; ?; Abalos et al., 2016). In this study, we propose a new framework to quan-
tify both symmetric and asymmetric mixing in a modified Lagrangian coordinate, and apply it to
synoptic eddy lifecycle simulations and a climate simulation.

As there are many ways to quantify isentropic mixing, there are many physical processes that
drive them. Among them, Rossby wave breaking (RWB) is an important one both within the
troposphere and across the tropopause. Near the tropopause, RWB has been shown by numerous studies to be associated with the exchange of mass or chemical tracers between the stratosphere and the troposphere (e.g. Appenzeller and Davies, 1992; Trepte et al., 1993; Chen, 1995; Appenzeller et al., 1996; Jing et al., 2004). Especially strong exchange occurs in the presence of a double tropopause (e.g. Randel et al., 2007; Pan et al., 2009) which is a feature related to RWB (e.g. Wang and Polvani, 2011). In the troposphere, we will show that RWB is also closely linked to isentropic mixing.

RWB can be classified into two distinct types – anticyclonic wave breaking (AWB) and cyclonic wave breaking (CWB), and the frequency of each is strongly coupled to the midlatitude jet stream. Climatologically, AWB occurs most frequently on the equatorward flank of the jet where the horizontal wind shear is anticyclonic, while CWB occurs most frequently on the poleward flank of the jet. As the jet shifts poleward, the total frequency of AWB increases whereas that of CWB decreases (e.g. Strong and Magnusdottir, 2008; Rivière, 2011; Barnes and Hartmann, 2012). This RWB-jet position relationship is especially important because the jet stream position is very responsive to both internal climate variability such as the El-Niño Southern Oscillation (ENSO) (e.g. Rasmusson and Wallace, 1983; Chen and van den Dool, 1999; Ren et al., 2008), and external climate forcings such as increasing greenhouse gas concentrations and ozone depletion (e.g. Thompson and Solomon, 2002; Butler et al., 2010; Barnes and Polvani, 2013). It is through such coupling that RWB is linked to climate variability on different time scales, and thus, can drive the variability of global transport and mixing of various chemical tracers.

Quantifying the isentropic mixing by RWB is thus key to understanding the variability of the climate-scale distribution of various chemical tracers. Polvani and Esler (2007) (hereafter PE07) quantified the combined symmetric and asymmetric mixing associated with the two types of RWB in idealized eddy lifecycle simulations. They found that CWB exhibits stronger mixing from the
stratosphere to the troposphere compared to AWB. Considering the coupling of the jet stream position and AWB vs. CWB frequency, such an difference has significant implications since it may translate into a change in total mixing when the jet stream shifts. In this work, we explore these implications by asking two questions: (1) Is there a robust difference in asymmetric and symmetric mixing efficiency between AWB and CWB? (2) How does this difference, if it exists, translate into a dependence of total mixing on jet variability?

The paper is organized as follows. A description of the numerical simulations is given in section 2. In section 3, we introduce a new framework to quantify both symmetric and asymmetric mixing in a modified Lagrangian coordinate. Section 4 revisits the idealized simulations of PE07 and demonstrates how our new framework can provide additional insight. Section 5 explores both types of mixing linked to RWB in a climate simulation, and tests the robustness of the difference between AWB and CWB. Section 6 shows how such difference can impact total mixing as the jet-stream shifts. Conclusions are given in section 7.

2. Model setups

All simulations in this study are performed by integrating a primitive equation model with no topography. The specific model used is the Geophysical Fluid Dynamics Laboratory (GFDL) spectral dry dynamical core with a horizontal resolution of T42, and an integrating time step of 1200 seconds. The model uses $\nabla^8$ horizontal hyperdiffusion for dynamical variables such as temperature, vorticity and divergence. There is no explicit vertical diffusion for any variable in this model.

With the same model setup described above, we carry out two different sets of simulations. The first set of simulations are idealized eddy lifecycle simulations with initialized passive tracers that follow PE07 (to be discussed in detail). In this simulation, we aim to validate our new framework
for quantifying isentropic mixing by applying it to stratosphere-troposphere-exchange (STE) in the identical setting used by PE07, and to gain additional insight. The second simulation is an idealized climate run based on Held and Suarez (1994). In this simulation we investigate the difference in the strength of AWB and CWB related mixing in an idealized climate setting, and use the results to explain the dependence of isentropic mixing on jet variability. The detailed configuration for these two simulations is described in the next two subsections.

a. Idealized eddy lifecycle simulation

Following PE07, we reproduce the two idealized eddy lifecycles (LC1 and LC2) first introduced by Thornicroft et al. (1993). The model is configured with 30 unevenly spaced vertical layers and no diabatic heating during the 20-day run. In both the LC1 and LC2 simulations, only the initial conditions are prescribed. For LC1, a baroclinically unstable jet is prescribed on day 1 and the eddies grow and break anticyclonically at the late stages of its lifecycle, which corresponds to AWB in the real atmosphere. The initial zonal wind is given by:

\[
\begin{align*}
    u_1(\phi, z) &= U_0 F(\phi) \left[\left(\frac{z}{z_T}\right)e^{-\left[(\frac{z}{z_T})^2 - 1\right]/2}\right] \\
    F(\phi) &= \begin{cases} 
        \sin(\pi(\sin\phi)^2) \theta & \text{for } \phi > 0 \\
        0 & \text{for } \phi < 0.
    \end{cases}
\end{align*}
\]

where \(z \equiv H \log(p_0/p)\) is the log-pressure height. For LC2, the initial zonal wind is obtained by adding a bottom-heavy cyclonic shear \(u_s\) to the initial wind profile for LC1,

\[
\begin{align*}
    u_2(\phi, z) &= u_1(\phi, z) + u_s(\phi, z) \\
    u_s(\phi, z) &= -U_s e^{-z/z_s} \sin(2\phi) \Delta_s \left[\frac{\phi - \phi_s}{\Delta_s}\right] e^{-\left[\frac{\phi - \phi_s}{\Delta_s}\right]^2}
\end{align*}
\]
In the LC2 simulation the eddies grow and break cyclonically at the late stages of the lifecycle which corresponds to CWB. The constants appearing in (1) and (2) are the same as those in PE07. For both lifecycles, the initial conditions for temperature and surface pressure are set to be in thermal wind balance with the zonal wind profiles prescribed in (1) and (2). See appendix A of PE07 for the additional details on calculating these balanced initial conditions. In addition to the balanced temperature profile, a wavenumber-6 perturbation centered at 45°N is added to the initial temperature field to give rise to the growing eddies.

For both the LC1 and LC2 simulations, the tracer setup is similar to PE07 in that we initialize two passive isentropic tracers: a stratospheric tracer (\(S\)) and a tropospheric tracer (\(T\)) on isentropes ranging from 290K to 380K. The initial tropopause is defined in PE07 as a nearly vertical boundary that partitions isentropes into a poleward half (stratosphere) and a equatorward half (troposphere) (see Fig. 5 in PE07). The initial concentrations of the two tracers are set such that poleward of the initial tropopause \(S = 1\) and \(T = 0\), while equatorward of it \(T = 1\) and \(S = 0\). Since the simulations are adiabatic, \(S + T = 1\) on isentropes from 290K to 380K at all times by construction. During the simulations, the tropopause is defined by an evolving boundary between the two tracers as the contour of \(S = T = 0.5\). Both \(S\) and \(T\) are spectral tracers and a \(\nabla^8\) hyperdiffusion is applied at each time step. We refer readers to PE07 for additional details.

\[b. \text{Idealized climate simulation}\]

We perform a climate simulation forced by the idealized diabatic heating scheme proposed by Held and Suarez (1994). Specifically, we set the diabatic heating as a relaxation to a prescribed zonally-symmetric equilibrium temperature profile:

\[
T_{eq} = \max\left\{ 200K, \left[ 315K - (\Delta T)_{y} \sin^{2}\phi - (\Delta \theta)_{z} \log \left( \frac{p}{p_0} \right) \cos^{2}\phi - \varepsilon \sin \phi \right] \left( \frac{p}{p_0} \right)^{\kappa} \right\}
\]  (3)
For this relaxation scheme, the individual terms in (3) are equivalent to constant heating or cooling terms. Idealized as it is, (3) includes the most essential elements of climate forcing – heating in low latitudes and cooling in high latitudes, as represented by term A. Term B is a modification for the tropics, as manifested by the $\cos^2\phi$ factor. Without term B, but with the inclusion of factor D, (3) would yield an atmosphere that was stratified dry adiabatically. Term B thus can be thought of as a representation of stabilization by moist convection in the tropics which is not represented explicitly in this model setup. Term C adds a hemispheric asymmetry to the equilibrium temperature that accounts for seasonality in radiative forcing. In this study we set $\varepsilon = 10$ which represents winter conditions in the Northern Hemisphere. We focus on winter because the eddy activity is strongest in winter when the baroclinicity is largest. In addition, the low frequency variability that modulates eddy activity (e.g. the annular mode) is also strongest in the winter.

While the equilibrium temperature profile given in (3) leads to a midlatitude circulation that is generally representative of that is observed, there are still noticeable differences, especially in the tropics. For example, the Hadley cell in this dry general circulation model (GCM) is substantially weaker than in observations. However, the main purpose of this simulation is not to reproduce the observed circulation exactly (there is no topography, for example), but to serve as a stepping stone between the unforced eddy lifecycle simulation and reality by removing the complexities of full GCMs that are not essential for isentropic mixing.

To study the isentropic mixing along isentropes, we initialize a spectral tracer that serves as our Lagrangian coordinate. The tracer can be thought of having a source in the boundary layer from 60°S to 60°N and a sink that includes the entire atmospheric column at the poles so that an equator-to-pole gradient in tracer concentration is maintained along isentropic surfaces. As will be explained in the next section, this monotonic meridional gradient of the tracer is required for our tracer coordinate system. The tracer concentration is set to 0 anywhere poleward of 85°N or
85°S, and
\[
\frac{1}{4.5} [3.7 - P_2(\sin \phi) - P_4(\sin \phi)]
\] (4)
in the boundary layer. $P_2$ and $P_4$ are the second and fourth order Legendre polynomials respectively. (4) is qualitatively similar to a tracer concentration of 1 from 60°S to 60°N and 0 elsewhere in the boundary layer, but with a gradual transition from 1 to 0 on the edges, which prevents spurious tracer concentration during integration due to Gibbs effect. The tracer is otherwise passive throughout the rest of the atmosphere. As in the lifecycle simulations, a $\nabla^8$ hyperdiffusion is used for the spectral tracer.

With the setup described above, the model is run with 20 evenly spaced sigma levels for 25 years. We use the last 22 years for analysis which allows for a spin-up period of about 1000 days. All of the variables are output every 6 hours.

3. Quantification of isentropic symmetric and asymmetric mixing

a. MLM framework and the tracer equation

Viewed in an Eulerian framework, the distribution of a passive tracer can be accounted for by two processes: advective fluxes and diffusive fluxes. The idea of a modified Lagrangian coordinate is to reformulate the Eulerian tracer equation in a flow-following two-dimensional coordinate so that reversible advective fluxes are absent. Specifically, we can choose potential temperature ($\theta$) as the vertical coordinate, and the concentration of a passive tracer ($q$) as the meridional coordinate. As shown by Nakamura (1995), under such a coordinate transformation, the mass continuity equation becomes:

\[
\frac{\partial \mathcal{M}(1)}{\partial t} = -\frac{\partial \mathcal{M}(\dot{q})}{\partial q} - \frac{\partial \mathcal{M}(\dot{\theta})}{\partial \theta}
\] (5)
where $\dot{q}$ and $\dot{\theta}$ are the material derivatives of $q$ and $\theta$, and

$$\mathcal{M}(\chi) \equiv \iint_{\chi \leq q} \sigma \, dA = \int_{\chi \leq q} dq^* \int_{\chi^* \leq q} \frac{\sigma}{|\nabla q^*|} \, dl \quad (6a)$$

and

$$\frac{\partial \mathcal{M}()} {\partial q} = \oint q \frac{\sigma}{|\nabla q^*|} \, dl \quad (6b)$$

in which $\sigma \equiv -g^{-1} \frac{\partial p}{\partial \theta}$ is the pseudo-density in isentropic coordinates. Note that in the climate simulation $q$ decreases with latitude as determined by the sources and sinks of our passive tracer. On the lhs of (5) is the mass tendency within a contour of $q$. The first term on the rhs of (5) accounts for horizontal transport of mass across a contour of $q$, while the second term represents the differential vertical transport of mass across isentropic surfaces. These transport terms are proportional to the magnitudes of $\dot{q}$ and $\dot{\theta}$ which represent tendencies due to non-conservative processes such as diffusion and diabatic heating. These non-conservative processes correspond to irreversible transport and mixing of an air mass. Conservative processes such as reversible advection are invisible to (5).

We can now write the MLM equation for the mass budget of a tracer $\chi$ within a contour of $q$:

$$\frac{\partial \mathcal{M}(\chi)} {\partial t} = -\frac{\partial \mathcal{M}(\dot{q}\chi)} {\partial q} - \frac{\partial \mathcal{M}(\dot{\theta}\chi)} {\partial \theta} + \mathcal{M}(\dot{\chi}) \quad (7)$$

and we refer the reader to Appendix A for the derivation. The first three terms are very similar to the three terms in (5). This connection is clearer if one thinks of $\chi$ as a ubiquitous tracer with a concentration of 100% ($\chi=1$) everywhere. In this special case, plugging $\chi=1$ into (7) reduces the tracer equation to the mass continuity equation (5), where $\mathcal{M}(\dot{\chi})$ vanishes since $\dot{\chi} = 0$. For a general tracer $\chi$, the first two terms on the rhs of (7) are mass flux weighted by the tracer concentration and represent the effective advection of $\chi$. As will be shown in the next subsection, this “advection” is due to asymmetric mixing on the two sides of the coordinate contour. The
physical meaning of the third term, \( \mathcal{M}(\dot{\chi}) \), is two-fold since \( \dot{\chi} \) can be partitioned into two parts:

\[
\dot{\chi} = \dot{\chi}_{\text{diff}} + \dot{\chi}_{\text{prod}}
\]

The first term is the tendency due to diffusion (\( \dot{\chi}_{\text{diff}} \)), and the second term is the tendency due to tracer production and loss (\( \dot{\chi}_{\text{prod}} \)), for example, due to chemical reaction or phase change. Due to the linearity of operator \( \mathcal{M}(\cdot) \), we can partition \( \mathcal{M}(\dot{\chi}) \) accordingly:

\[
\mathcal{M}(\dot{\chi}) = \mathcal{M}(\dot{\chi}_{\text{diff}}) + \mathcal{M}(\dot{\chi}_{\text{prod}})
\]

The physical meaning of \( \mathcal{M}(\dot{\chi}_{\text{prod}}) \) is thus the change in the tracer mass of \( \chi \) within a tracer contour of \( q \) due to the area-integrated production within the \( q \) contour. The reason the production does not show up in (5) is simply that mass cannot be generated or destroyed. In this study, we focus on the dynamic aspect of equation (7) and thus will only use a passive tracer so that \( \mathcal{M}(\dot{\chi}_{\text{prod}}) = 0 \) at all times. The physical meaning of \( \mathcal{M}(\dot{\chi}_{\text{diff}}) \) is less straightforward and represents the symmetric mixing of tracer \( \chi \) into a \( q \) contour by swapping mass across the contour. That is, although the air is refreshed on both sides, the amount of mass on either side remains constant, and thus, total mass flux is zero which is why it does not appear in the mass continuity equation (5). We will further justify the interpretation of this term as symmetric mixing in the following subsection.

b. Isentropic asymmetric and symmetric mixing

The interpretation of the first and third terms on the rhs of equation (7) is most clear if we let \( \chi = q \), namely, we use the tracer of interest as our meridional coordinate. In this case, we have an equation for the mass budget of \( q \) within a contour of \( q \):

\[
\frac{\partial \mathcal{M}(q)}{\partial t} = -\frac{\partial \mathcal{M}(\dot{q})}{\partial q} q - \frac{\partial \mathcal{M}(\dot{\theta}q)}{\partial \theta} + \mathcal{M}(\dot{q})
\]

(9)
where we have used the fact that:

\[
\frac{\partial \mathcal{M}(\dot{q}q)}{\partial q} = \oint_q \frac{\dot{q}^* q^*}{|\nabla q^*|} \sigma dl = \frac{\dot{q}^* q^*}{|\nabla q^*|} \sigma dl \cdot q = \frac{\partial \mathcal{M}(\dot{q})}{\partial q} q
\]

The third term on the rhs of (9) is termed symmetric mixing in this study. It appears as an area integral in (9) but can be thought of as a line integral if \( \dot{q} \) takes a diffusive form such as:

\[
\dot{q} = \kappa \nabla^2 q = \kappa \nabla \theta \cdot \nabla q
\]

Plugging (10) into the term \( \mathcal{M}(\dot{q}) \) and applying the divergence theorem leads to

\[
\mathcal{M}(\dot{q}) = \kappa \oint_q |\nabla q^*| \sigma dl
\]

The magnitude of \( \mathcal{M}(\dot{q}) \) is thus proportional to the gradient of the tracer \( q \) across the contour.

In the case of non-divergent barotropic flow, \( \sigma \) is a constant \( (\sigma = \sigma_0) \) and we can rewrite \( \mathcal{M}(\dot{q}) \) in a diffusive form through application of (6b):

\[
\mathcal{M}(\dot{q}) = \sigma_0 \kappa_{eff} \frac{\partial q}{\partial A}
\]

where \( A(q, \theta, t) \) is the area enclosed by a contour \( q \), and

\[
\kappa_{eff} = \kappa \frac{\partial}{\partial A} \int \int |\nabla q|^2 dA / (\frac{\partial q}{\partial A})^2
\]

is identical to the effective diffusivity proposed by Nakamura (1996). (See Appendix B for the derivation.) The fact that we can write \( \mathcal{M}(\dot{q}) \) as an effective diffusivity times the tracer gradient further justifies our interpretation of this term as representing symmetric mixing. Note that the effective diffusivity \( \kappa_{eff} \) in the form of (13) is positive definite so the sign of the mixing is determined only by the tracer gradient in this instance.

The first term on the rhs of (9) is the product of the meridional net mass flux, \(-\partial \mathcal{M}(\dot{q})/\partial q\), and the tracer concentration \( q \). The interpretation of the mass flux term is two-fold. On the one hand, it balances the vertical diabatic mass flux, \(-\partial \mathcal{M}(\dot{q})/\partial \theta\), in the climatological mean and thus is part...
of the diabatic circulation. Leibensperger and Plumb (2014) use the diabatic mass flux to define a meridional advection term in their MLM tracer equation, which takes an advection-diffusion form. In light of this, the $-q \partial \mathcal{M}(\dot{q}) / \partial q$ term has the effect of “advection” in our Lagrangian coordinate. On the other hand, the meridional mass flux arises from asymmetric isentropic mixing. This is mathematically evident since the mass flux is the meridional gradient of the symmetric mixing term $\mathcal{M}(\dot{q})$ as shown in (9). Physically, mixing acts to reduce the gradient of a passive tracer and pushes the tracer contours farther away from each other. For a particular tracer contour, if the strength of symmetric mixing is not equal on its two sides, the tracer contour will be pushed away from the region of stronger mixing, accompanied by a change in its latitude. Alternatively one can visualize this as an increase in air mass on the side of the tracer contour with stronger mixing.

Based on the discussion above, we refer to the first term on the rhs of (9) as asymmetric mixing, and the third term as symmetric mixing. As has been discussed and will be shown, the two have very different behaviors. The asymmetric mixing is directional and its direction is dictated by the mass flux. The symmetric mixing is diffusive and its direction is down-gradient.

c. Calculation of asymmetric and symmetric mixing

To calculate the asymmetric and symmetric mixing terms in (9), we need to first determine the tracer contours to use as coordinates. In the idealized lifecycle simulations, the only contour used is the 0.5 contour for both $S$ and $T$ tracers which defines the tropopause. For the climate simulation, we pick 32 contours of $q$ for each isentropic level in the Northern Hemisphere. What follows describes how these 32 contours are chosen. We determine these 32 values of $q$ by the one-to-one relationship between $q$ and equivalent latitude $\phi_e$ (Butchart and Remsberg, 1986) on
each isentrope and at each time step:

$$\phi_e(q) = \arcsin \left[ 1 - \frac{A(q)}{2\pi a^2} \right]$$  \hspace{1cm} (14)

Specifically, we pick 32 values of $\phi_e$ and then obtain the 32 values of $q(\phi_e)$ by reversing the relationship of (14) at each time step. Finally, we temporally average $q(\phi_e)$ over a year of simulation and use these averaged $q(\phi_e)$ as the meridional coordinate. All of the MLM mixing results, however, are displayed with the corresponding equivalent latitude ($\phi_e$) along the x-axis.

The terms in (9) are calculated as area integrals of combinations of $\sigma$ and $\dot{q}$. $\dot{q}$ is calculated as the 4th order hyperdiffusion in the model, and we output $\dot{q}$ at 6-hourly intervals and interpolate it and $\sigma$ on isentropic surfaces following the interpolation scheme by Edouard et al. (1997). The integration area for the numerator in the asymmetric mixing term, $\partial M(\dot{q})$, is shown in Fig. ??a as a thin green band. The denominator $\partial q$ is calculated as the difference $q_1 - q_2$ of the two contours that bound the green area. The integration area for the symmetric mixing term, $M(\dot{q})$, is illustrated in Fig. ??b. For the climate simulation, we are especially interested in mixing driven by eddies. To serve this purpose, we partition the integration for $M(\dot{q})$ into two parts following the idea of Chen and Plumb (2014):

$$M(\dot{q}) = \overline{M}(\dot{q}) + \Delta M(\dot{q})$$  \hspace{1cm} (15)

The first term is $\overline{M}(\dot{q})$ which represents the integration poleward of an equivalent latitude circle ($\phi_e$). The integration area for this term is indicated by the black dashed circle in Figs. ??b-d. This term is driven only by diffusion. The second term is $\Delta M(\dot{q})$ which is driven by zonal asymmetries in the flow (i.e. eddies). The integration area of this term is indicated by the shading in Fig ??d where the red area is weighted by 1 and the blue area is weighted by $-1$. $\Delta M(\dot{q})$ captures symmetric mixing by eddies (from now on referred to as eddy mixing) and is the component we focus on here.
Although in this study we focus on idealized settings where only passive tracers are used, (9) can be applied to more realistic and sophisticated tracers in GCMs if diffusion is calculated explicitly. In that case, using (9) along with (8) one can separate the contributions from asymmetric, symmetric mixing and chemical production and loss to the distribution of a species.

\textit{d. Linking asymmetric and symmetric mixing to Rossby wave breaking}

In this study, we separate mixing by the overturning direction of Rossby wave breaking – anticyclonic and cyclonic – due to the robust dynamical relationship between the two types of wave breaking and the jet streams (e.g. Barnes and Hartmann, 2012). Note that Nakamura (2004) separates asymmetric mixing into a poleward and an equatorward component. Since cyclonic and anticyclonic wave breaking events are comprised of a combination of poleward and equatorward mixing, our approach is distinct from this earlier work. For our idealized lifecycle simulations, linking asymmetric and symmetric mixing to the two types of wave breaking is straightforward. As in PE07, we assume all of the mixing in the LC1 simulation is linked to AWB while all of the mixing in the LC2 is linked to CWB. However, for the climate simulation in which AWB and CWB coexist every day along with other non-breaking waves, we need a dedicated algorithm to separate the contribution from the two types of wave breaking from all of the other processes. To serve this purpose, we first need to detect the two types of RWB in the model output. Unlike some previous studies where contours of potential vorticity (PV) on isentropes are used to identify wave breaking events (e.g. Strong and Magnusdottir, 2008; Liu et al., 2014; Liu and Barnes, 2015), we base our algorithm on the tracer contours themselves. In fact, the contours of the tracer $q$ in our climate simulation align well with PV contours on isentropic surfaces and thus our method is very similar to that of previous studies (not shown). The reason we favor $q$ over PV for RWB detection
is that our quantification of the mixing is based on \( q \) contours already. Thus, using \( q \) to detect RWB is natural for linking mixing to RWB events in our framework.

Specifically, we detect RWB based on wave lobes illustrated in Fig ??c and ??d. If the left-most/rightmost point of one lobe is to the left/right of its base line (shown as black dashed segments), then we identify the lobe as overturning. We require the extent of overturning be at least 5.5° of longitude to exclude small scale disturbances that are not linked to large-scale RWB. Then, according to the direction of overturning, we partition the overturning lobes into AWB and CWB. For example, in Fig ??d the overturning lobe is identified as CWB and marked by dark blue shading. The eddy mixing \( \Delta M(q) \) associated with this CWB lobe is the integration over the dark blue area in Fig ??d. Similarly, the asymmetric mixing associated with the CWB lobe is the part of the integration that overlaps with the lobe, as illustrated by dark green region in Fig ??c. The frequency of occurrence of RWB is defined as the ratio of RWB area to the total wave area (e.g. dark blue versus all shading in Fig ??d). We compared the climatology of RWB frequency obtained by this method with that obtained by an algorithm using PV contours (Liu et al., 2014; Liu and Barnes, 2015). They agree well with each other (not shown), thus further supporting our use of \( q \) for identifying RWB in this study.

A caveat should be raised about local RWB contribution to symmetric mixing. Since the divergence theorem does not hold for local integration, identities (11) and (12) do not hold for the RWB contribution. In other words, the local contribution to symmetric mixing is not guaranteed to be down-gradient although the total symmetric mixing is, provided that the diffusion takes the form of (10).
4. Idealized lifecycle simulations

With the new framework to quantify asymmetric and symmetric mixing introduced in the last section, we aim to achieve two goals in this section: (1) validate our new framework for quantifying the two components of total mixing, and (2) gain additional insight into the stratosphere-troposphere-exchange (STE) in the idealized experiments explored by PE07.

Figure ?? shows snapshots of tracer $S$ on the 335K isentrope in the two idealized lifecycles, as also shown by PE07. Orange indicates the stratosphere and blue indicates the troposphere. On day 8, the anticyclonic overturning in LC1 is depicted by the blue shading equatorward of the tropopause (denoted by the black dashed line), and the cyclonic overturning in LC2 is depicted by orange shading poleward of the tropopause (Fig. ??c and ??d respectively). It is evident in Fig. ??c and ??d that CWB in LC2 is much more active in stretching and overturning the tropopause compared to AWB in LC1. As a result, there is a larger interface between the $S$ and $T$ tracers for mixing to work on for CWB. Consistent with this picture, PE07 documented 50% stronger stratosphere-to-troposphere combined mixing in LC2 than in LC1.

Figure ?? shows the detailed quantification of STE in the two idealized lifecycles. Dashed blue lines in the upper row shows the troposphere-to-stratosphere combined mixing quantified using PE07’s method. Specifically, we show the evolution of the mass of tracer $T$ poleward of the tropopause (0.5 contour) in LC1 (left) and LC2 (right) simulations. This corresponds to the time integration of the $\frac{\partial \mathcal{M}(q)}{\partial t}$ term in (9) poleward of $q = 0.5$, with $q = T$. Positive slope of the dashed blue lines indicate an increase of $T$ in the stratosphere, or a poleward combined mixing of tracer $T$ into the stratosphere. Similarly, dashed blue lines in the bottom row of Fig. ?? show the stratosphere-to-troposphere combined mixing by integrating the mass of $S$ poleward of the 0.5 contour. Negative slopes of the blue lines indicate a decrease of $S$ in the stratosphere, or an
equatorward combined mixing of tracer $S$ into the troposphere. The dashed blue lines in Fig. ?? agree qualitatively with PE07 – LC2 is comparable to LC1 in terms of troposphere-to-stratosphere combined mixing but has much stronger stratosphere-to-troposphere combined mixing compared to LC1.

So far we have only looked at the combined mixing in these lifecycle simulations. With our new framework, we can now quantify the asymmetric and symmetric components separately. The time integration of asymmetric mixing is depicted as green lines while that of symmetric mixing is depicted as red lines in Fig. ?? and the red lines (which indicate symmetric mixing) are the exact opposite for $q = T$ and $q = S$. This is due to the fact that $S = 1 - T$, or in other words, the two tracers constitute the whole air mass. As a result, $S$ and $T$ have gradients of the same magnitude but in opposite directions. Since the symmetric mixing is determined by the gradient of the tracer, opposite gradients give rise to opposite mixing. Another feature that stands out in Fig. ?? is that the symmetric mixing (red lines) exhibits a
monotonic change with time. This is because symmetric mixing is usually down-gradient and the
gradient of the tracer is fixed to be the same sign by construction throughout the entire simulation.
Specifically, the down-gradient direction for tracer $T$ is always poleward and thus the symmetric
mixing flux of $T$ is poleward (positive; Figs. ??a and ??b), while the down-gradient direction for
tracer $S$ is equatorward and the symmetric mixing flux of $S$ is thus always equatorward (negative;
Figs. ??c and ??d). Note that the meaning of asymmetric and symmetric mixing is very straight-
forward in this setup. If there is no asymmetric mixing (green lines), the mixing flux of the $S$ and
$T$ tracers (blue lines) would be equal but opposite, meaning the amount of $S$ tracer mixed into the
troposphere would be the same as the amount of $T$ tracer mixed into the stratosphere.

If we compare the magnitude of asymmetric and symmetric mixing terms for LC1 and LC2 in
Fig. ??, we see that LC2 has both stronger asymmetric and symmetric mixing compared to LC1,
consistent with what one might expect from Fig. ?? In both LC1 and LC2, the direction of mass
flux is equatorward from the stratosphere to the troposphere (negative green lines in Figs. ??a-d),
meaning the troposphere is expanding and the stratosphere is losing mass due to asymmetric mixing
across the tropopause. For the tropospheric tracer $T$, this means that the direction of asymmetric
mixing opposes the direction of symmetric mixing (Figs. ??a and ??b). Before day 4, poleward
symmetric mixing of tracer $T$ into the stratosphere dominates over the asymmetric mixing due to
the large initial gradient of $T$. After day 4, the stratosphere has been loaded with some $T$ tracer
and the equatorward mass flux by asymmetric mixing starts to bring some of this $T$ tracer back in
to the troposphere. As a result, although LC2 exhibits stronger asymmetric and symmetric mixing,
the net change in mass of $T$ in the stratosphere ends up being similar to that of LC1 (blue lines
in Figs. ??a and ??b). For the stratospheric tracer $S$, the directions of asymmetric and symmetric
mixing are the same (Figs. ??c and ??d). After day 4, as equatorward symmetric mixing increases
$S$ in the troposphere, the equatorward asymmetric mixing brings even more $S$ tracer from the
stratosphere to the troposphere. As a result, LC2, with both stronger asymmetric and symmetric mixing, exhibits significantly stronger combined mixing compared to LC1.

Section summary

We applied our new framework to quantify MLM asymmetric and symmetric mixing in the idealized lifecycle simulations of PE07. The results of this analysis can be summarized by the following two conclusions:

- the new framework is able to close the budget of tracer mass in both LC1 and LC2.
- CWB has both stronger asymmetric and symmetric mixing than AWB across an idealized tropopause.

In the next section, we show how the second result (and others) manifest in a full climate simulation.

5. Climatology of asymmetric and symmetric mixing

Figure ?? shows the climatology of potential temperature (solid line) in pressure coordinates from the idealized climate simulation. The shading denotes the range of isentropes (270K-350K) on which we quantify mixing of the idealized tracer. We further divide the shaded area into the middleworld (green shading, 300K-350K) and the underworld (pink shading, 270K-295K) (e.g. Shaw and Austin, 1930; Hoskins, 1991). The underworld is defined by isentropes that are always below the tropopause denoted by a black dashed line, while the middleworld is defined by isentropes that cross the tropopause. In the rest of the paper, the majority of the results are shown on these isentropes or as an average over the underworld or the middleworld.

As shown by equation (9), both asymmetric and symmetric mixing are influenced by the tracer distribution. Since the tracer setup here does not represent any specific chemical tracer in the real
atmosphere, it would not be generally relevant to show its mixing. Fortunately, we can readily extract information more applicable to general situations by calculating the net mass flux (rather than the tracer mass flux) resulting from asymmetric mixing and calculating the eddy diffusivity to quantify symmetric mixing strength. By doing so, we factor out the influence of the specific tracer distribution in our simulation and obtain a purely dynamical measurement of the strength of isentropic mixing.

The net mass flux is calculated as \( \frac{\partial \mathcal{M}(\dot{q})}{\partial q} \) in the asymmetric mixing term in (9). The eddy diffusivity is calculated as the eddy mixing term \( \Delta \mathcal{M}(\dot{q}) \) from (15) normalized by the tracer gradient and mean mass density, that is, \( \frac{\Delta \mathcal{M}(\dot{q})}{(\overline{\sigma} \frac{\partial q}{\partial \phi})} \), where \( \overline{\sigma} \) is the average pseudo-density over a tracer contour in climatological mean. Note that this expression of eddy diffusivity is very similar to that of Chen and Plumb (2014) which employs a hybrid Eulerian-Lagrangian mean coordinate. The difference is that they factor out the Eulerian zonal average of \( \sigma \) whereas we factor out the Lagrangian average of \( \sigma \). In the rest of the paper, we use mass flux to represent asymmetric mixing strength and eddy diffusivity to represent symmetric mixing strength.

Note that although many studies use equivalent latitude as the coordinate, we use a tracer coordinate throughout this study. However, we show our results as a function of approximate equivalent latitude of the tracer contours to provide a geographical reference. We refer readers to Section 3c for how we obtain the approximate equivalent latitudes.

The spatial pattern of the climatological eddy diffusivity is shown in Fig. ??a as a function of approximate equivalent latitude of tracer contours, along with the jet stream (black solid contour). In Fig ??a, eddy diffusivity minimize at the jet core near the tropopause, indicating that jet acts as a mixing barrier (e.g. Haynes and Shuckburgh, 2000b; Chen and Plumb, 2014). The eddy diffusivity linked to RWB bears a very similar spatial pattern to the total (Fig. ??b), and in this model about 50% of the eddy symmetric mixing is done by large-scale RWB. The RWB frequency
of occurrence is plotted in grey in Fig. ??b, and agrees well with the magnitude of the eddy diffusivity, with the largest eddy diffusivities overlapping the maxima of RWB frequencies.

Figures ??c-d show the spatial patterns of the climatological net mass flux due to asymmetric mixing and the contribution from RWB. As discussed in Section 3b, the net mass flux due to asymmetric mixing should go toward the region of strongest symmetric mixing. Such relationship between symmetric and asymmetric mixing can be seen by comparing the eddy diffusivity and mass flux in Fig ??c. The diffusivity minimizes around 40°N near the tropopause, increasing equatorward and poleward until 60°N (Fig ??a). Correspondingly, the mass flux diverges away from 40°N near the tropopause and the poleward flux stops at 60°N (Fig ??c). The large magnitude of the equatorward mass flux in the subtropics between 310K and 320K is partly due to the fact that the mass represented by these few isentropes is very large (Fig. ??). Poleward of 60°N, the diffusivity decreases with latitude and the mass is hence drawn equatorward (negative mass flux in Figs ??a and ??c). Figure ??d shows the contribution by RWB to the mass flux (shading), which resembles the total mass flux in most regions. The only exception occurs near the jet core where total mass flux is poleward while RWB mass flux is absent.

The RWB contribution to eddy diffusivity and mass flux is further divided into AWB contribution and CWB contribution shown in Fig ?? (shading) as a function of approximate equivalent latitude, along with their corresponding frequency of occurrence (grey contours). AWB mainly occurs at lower latitudes equatorward of the jet where the mean horizontal shear of wind is anticyclonic (Figs. ??a and ??c), while CWB mainly occurs at higher latitudes poleward of the jet where the mean shear is cyclonic (Figs. ??b and ??d). Overall, the contribution of AWB and CWB to eddy diffusivity and mass flux is proportional to their frequency of occurrence, but clearly eddy diffusivity is better coupled to wave breaking frequency.
A true comparison of the mass flux and eddy diffusivity between the two types of RWB can be obtained by dividing their climatological mean values by the frequency of RWB occurrence. Through this normalization, we factor out the contribution of frequency and compare the mass flux and eddy diffusivity per occurrence of AWB/CWB, i.e., their efficiency. In the most of the underworld, AWB and CWB related mass flux has the same sign and comparable efficiency (not shown), that is, a single occurrence causes the same mass flux. In the middleworld between 45°N and 60°N, AWB and CWB have opposite signed mass flux. Specifically, between 45°N and 60°N, AWB exhibits equatorward mass flux and CWB exhibits poleward mass flux across the tropopause (Figs. ??c and ??d).

The comparison of eddy diffusivity per occurrence for the two types of RWB is shown in Figs. ??c and ??d as a function of approximate equivalent latitude, along with their counterparts in the idealized eddy lifecycle shown in Figs. ??a and ??b. In both the lifecycle and climate simulations, CWB (LC2) exhibits a larger diffusivity than AWB (LC1) in the midlatitudes. The difference is especially evident in the climate simulation near 50°N equivalent latitude (Figs. ??c and ??d). Recall that the lifecycle simulations are only made of one event, while the climate simulation is composed of thousands. Thus, we do not expect the results to be identical. However, the lifecycle simulations bear much resemblance to the climate simulations in Fig. ??, indicating the relevance of the idealized lifecycle simulations to more realistic climate settings. Interestingly, they also agree on the existence of negative eddy diffusivity equatorward of the jet stream in the subtropics linked to CWB (Figs. ??b and ??d). Note such negative diffusivity might be an artifact arising from the hyperdiffusion used in our model. In addition, as discussed in Section 3d, the negative diffusivity in Fig. ??d could also arise from the fact that local contribution to the eddy diffusivity is not guaranteed to be positive. Nevertheless, such up-gradient mixing behavior has been documented in reanalysis data within a similar latitude range (Birner et al., 2013), and more
study is needed to tell whether the negative eddy diffusivity linked to CWB in Fig. 7 is an artifact or has physical implications.

Section summary

We quantify the climatology of asymmetric and symmetric mixing, quantified by net mass flux and eddy diffusivity in a climate simulation and find,

• CWB has a comparable asymmetric mixing efficiency (net mass flux) but significantly stronger symmetric mixing efficiency (eddy diffusivity) than AWB in the underworld,

• the eddy diffusivity in lifecycle simulations bears resemblance to that in the climate simulations.

In the next section, we show how such differences in RWB symmetric mixing efficiency translate into a dependence of total symmetric mixing on jet variability.

6. Dependence on jet stream variability

The dependence of asymmetric and symmetric mixing on the internal variability of the jet, specifically the latitudinal shifts of jet, is explored by compositing the mass flux and eddy diffusivity with respect to various jet latitudes. We calculate the jet latitude as the latitude of maximum 850hPa zonal-mean zonal wind at 6-hour resolution (e.g. Barnes and Hartmann, 2010; ?). We group the jet latitudes into four bins centered at 39°N, 43°N, 47°N, and 51°N. These latitudes are chosen to make each bin have a similar sample size. We then composite RWB frequency, total mass flux and eddy diffusivity, and the contribution by RWB according to these jet latitude bins.
a. Tropospheric asymmetric and symmetric mixing

The tropospheric asymmetric and symmetric mixing are investigated by focusing on the underworld, which constitutes a corridor between the midlatitude boundary layer and the polar middle-to-upper troposphere (Fig. ??). The average frequency of RWB in this corridor is shown in Fig. ?? as a function of approximate equivalent latitude. Consistent with previous studies, AWB occurs more frequently while CWB occurs less frequently as the jet moves poleward. The extent of the frequency change associated with AWB and CWB are comparable in midlatitudes around 45°N. This indicates the compensation between the two types of RWB when jet shifts around, keeping RWB frequency relatively constant in the midlatitudes (Fig. ??c).

Figure ?? shows the composite mass flux due to asymmetric mixing and the contribution from RWB as a function of approximate equivalent latitude for various jet latitudes. RWB contributes less than 20% of the total mass flux in the subtropics and about 50% poleward of 55°N equivalent latitude. Neither total mass flux nor RWB contribution show significant change when the jet wobbles around (Figs. ??a and ??b). At least for the RWB contribution, this is due to the opposing changes in frequency of AWB and CWB (Figs. ??a and ??b), combined with the fact that the two types have comparable asymmetric mixing efficiencies in the underworld. As a result, the change in mass flux associated with AWB and CWB largely cancel each other (Figs. ??c and ??d) and leave the RWB mass flux largely unchanged (Fig. ??b). In this case, the mass flux in the underworld does not rely heavily on the jet latitude.

Figures ??a and ??b show the total eddy diffusivity and the contribution from RWB as a function of approximate equivalent latitude. RWB contributes slightly more than half of the total and both the total and RWB contribution to eddy diffusivity decrease when the jet is more poleward, in contrast to the case of mass flux. The difference lies in the fact that CWB has a significantly
stronger eddy diffusivity than AWB (Figs. ??c and ??d). As a result, the decrease in CWB diffusivity overwhelms the increase in AWB diffusivity although their changes in the frequency of occurrence are comparable (Fig. ??). This effect is most evident in the midlatitudes around $45^\circ N$ equivalent latitude. In this case, the difference between AWB and CWB symmetric mixing efficiency is key to determining the change in total eddy diffusivity linked with changes in the jet position.

b. Stratosphere-troposphere exchange (STE)

As shown in Fig. ??, RWB promotes STE by distorting the tropopause. The vertical distortion of the tropopause by AWB and CWB in the idealized lifecycle simulations is demonstrated in Fig. ?? by snapshots on day 6 and day 7 respectively. The two dates are chosen for the peak stage of AWB and CWB mixing (Fig. ??). In Fig. ??, both AWB and CWB are associated with an intrusion of stratospheric air that extends as low as 850hPa. These streamers are also observed in nature (e.g. Shapiro, 1980) and are responsible for injecting anthropogenic pollutants into the stratosphere and bringing ozone-rich stratospheric air down to the boundary layer. Quantifying the variability of RWB-related STE as shown in Fig. ?? is hence of great interest. To serve this purpose, the tropopause is first defined as the tracer contour whose mean position is closest to the 2PVU surface for the four jet latitudes. Then composite mass flux and eddy diffusivity are extracted along the corresponding tropopauses for the four jet latitudes.

The composite frequencies of occurrence of the two types of RWB in the middleworld are shown in Fig. ?? with respect to jet latitude. AWB mainly occurs equatorward of the jet while CWB mainly occurs poleward of the jet and there is little overlap. When the jet shifts poleward, AWB occurs more frequently while CWB occurs less frequently. Figures ??a and ??b show the total mass flux across the tropopause due to asymmetric mixing and the contribution from RWB. Both
the total and RWB contribution exhibit troposphere-to-stratosphere mass flux equatorward of 55°N equivalent latitude and stratosphere-to-troposphere mass flux poleward of 55°N. In terms of the variability associated with the jet shifting, a substantial portion of the total can be explained by the RWB contribution (compare Figs. ??a and ??b), which in turn, is dominated by the CWB contribution (Fig. ??c). Such dominance is largely due to the stronger decrease in CWB frequency compared to the increase in AWB frequency (Fig. ??). As such, the total mass flux across the tropopause shifts poleward and weakens as the jet shifts poleward. Due to the same line of reasoning, the variability of eddy diffusivity along the tropopause is also dominated by the changes in CWB (Fig. ??). The maximum eddy diffusivity moves poleward and weakens as the jet shifts poleward. While AWB and CWB do exhibit different symmetric mixing efficiencies, these changes are predominantly explained by changes in RWB frequency itself.

c. Section summary

As the jet shifts poleward,

- underworld mass flux due to asymmetric mixing does not change due to the compensating frequency changes of AWB and CWB, and their similar asymmetric mixing efficiencies.

- underworld eddy diffusivity (symmetric mixing strength) decreases due to the difference in AWB and CWB symmetric mixing efficiencies.

- across the tropopause, both mass flux due to asymmetric mixing and eddy diffusivity that represents symmetric mixing decrease due to the dominant decrease in CWB frequency.
7. Conclusions

A new framework to quantify both asymmetric and symmetric isentropic mixing is proposed and applied to Rossby wave breaking (RWB) in both free eddy lifecycle simulations and a forced climate simulation. The focus of this study is on the difference between the two flavors of RWB—anticyclonic wave breaking (AWB) and cyclonic wave breaking (CWB). It is found that in both sets of simulations and throughout different regions of the atmosphere, CWB has either comparable or stronger isentropic mixing than AWB. This result has implications for the influence of climate variability on isentropic mixing. Since CWB occurs less frequently and AWB occurs more frequently as the jet shifts poleward, mixing differences translate into a decrease in symmetric mixing strength (eddy diffusivity) within the troposphere, and decrease in both symmetric and asymmetric mixing (net mass flux) across the tropopause, as the jet shifts poleward.

Although the simulations in this study are highly idealized, we argue that the conclusions obtained are applicable to the real atmosphere. Abalos et al. (2016) used ERA-Interim to show that in winter the effective diffusivity across the tropopause decreases in the mid-to-high latitudes during a positive Northern Annular Mode when the jet is more poleward, and increases during positive ENSO when the jet is more equatorward. These results are consistent with the modulation of the jet stream on the two types of RWB. Although our study does not directly address the jet shift in response to external forcing (e.g. anthropogenic climate change), we expect similar results to what is found here due to the same relationship between RWB and the jet stream in climate change simulations (e.g. Barnes and Polvani, 2013). In fact, Orbe et al. (2015) demonstrated that isentropic mixing from the midlatitude boundary layer to the Arctic increases over the northeast Pacific with climate change in a state-of-the-art GCM where the jet stream shifts equatorward. This is con-
sistent with our argument that an equatorward jet shift increases the frequency of cyclonic wave breaking, and thus, increases the high-latitude mixing strength.

In the real atmosphere, certain constituents whose spatial distributions are subject to climate change, also contribute to the climate forcing. Examples include carbon dioxide, ozone, various species of aerosols and water vapor. Understanding the passive response of atmospheric constituents to climate change is thus an important step towards understanding the entire feedback loop. This study provides a quantification methodology and a mechanistic explanation for how the isentropic mixing of constituents may respond to future jet shifts.

**Acknowledgments.** This work was supported by the Climate and Large-scale Dynamics Program of the National Science Foundation under Grant 1419818.

**APPENDIX A**

**Derivation of modified Lagrangian mean (MLM) tracer equation**

Taking inspiration from Nakamura (1995), we show that the derivation of MLM tracer equation is to a large extent a mathematical transformation that starts with:

\[
\frac{\partial M(\chi)}{\partial t} = \frac{\partial}{\partial t} \int \int_{q^{*} \leq q} \sigma \chi dA = \int \int_{q^{*} \leq q} \frac{\partial (\sigma \chi)}{\partial t} dA - \frac{\partial}{\partial q} \int \int_{q^{*} \leq q} \sigma \chi \frac{\partial q^{*}}{\partial t} dA
\]

(A1)

The only physics that comes in is that of mass conservation:

\[
\frac{\partial (\sigma \chi)}{\partial t} = -\nabla_\theta \cdot (\sigma \chi u) - \frac{\partial (\sigma \chi \dot{\theta})}{\partial \theta} + \sigma \dot{\chi}
\]

(A2)

\[
\frac{\partial q^{*}}{\partial t} = -u \cdot \nabla_\theta q^{*} - \dot{\theta} \frac{\partial q^{*}}{\partial \theta} + q^{*}
\]

(A3)
where \( \dot{\chi} \) and \( \dot{q}^* \) represent non-conservative processes such as diffusion and/or production of the two tracers. Substituting (A2) and (A3) into (A1) and re-arranging,

\[
\frac{\partial \mathcal{M}(\chi)}{\partial t} = - \int \nabla_\theta \cdot (\sigma \chi \mathbf{u}) \, dA + \frac{\partial}{\partial q} \int \sigma \chi \mathbf{u} \cdot \nabla q^* \, dA + \frac{\partial}{\partial \theta} \int \sigma \dot{q}^* \, dA - \int \frac{\partial (\sigma \dot{\chi})}{\partial \theta} \, dA - \frac{\partial}{\partial q} \int \sigma \chi \dot{q} \, dA + \int \sigma \dot{\chi} \, dA
\]

(A4)

Applying the divergence theorem to the first term on the rhs of (A4) and using the relation (6b) leads to:

\[
- \int \nabla_\theta \cdot (\sigma \chi \mathbf{u}) \, dA = - \int_{q^* = q} \sigma \chi \mathbf{u} \cdot \nabla q^* \, dl = - \frac{\partial}{\partial q} \int \sigma \chi \mathbf{u} \cdot \nabla_\theta q \, dA
\]

(A5)

Substituting (A5) into (A4) and noting that:

\[
\frac{\partial}{\partial q} \int \sigma \chi \dot{q^*} \, dA - \frac{\partial}{\partial \theta} \int \sigma \dot{q} \, dA - \frac{\partial}{\partial \theta} \int \sigma \dot{\chi} \, dA
\]

(A6)

one obtains:

\[
\frac{\partial \mathcal{M}(\chi)}{\partial t} = - \frac{\partial}{\partial q} \int \sigma \chi \dot{q^*} \, dA - \frac{\partial}{\partial \theta} \int \sigma \dot{q} \, dA + \int \sigma \dot{\chi} \, dA
\]

\[
= - \frac{\partial \mathcal{M}(\dot{q})}{\partial q} - \frac{\partial \mathcal{M}(\dot{\theta} \chi)}{\partial \theta} + \mathcal{M}(\dot{\chi})
\]

(A7)

APPENDIX B

Derivation of effective diffusivity

Instead of assuming (10), we assume a more general form of hyperdiffusion for 2-D flow:

\[
\dot{q} = \kappa \nabla^{2m} \dot{q} = \kappa \nabla \cdot (\nabla^{2(m-1)} \dot{q})
\]

(B1)

where (10) is the special case in which \( m = 1 \). Plugging (B1) into \( \mathcal{M}(\dot{q}) \) leads to:

\[
\mathcal{M}(\dot{q}) = \sigma_0 \kappa \int \nabla \cdot (\nabla^{2(m-1)} \dot{q^*} \, dA
\]

31
Using the divergence theorem,

\[ \mathcal{M}(\dot{q}) = \sigma_0 \kappa \oint_{q^*} \nabla^2 (m-1) q^* \cdot \frac{\nabla q^*}{|\nabla q^*|} \, dl \]

and using relation (6b):

\[ \mathcal{M}(\dot{q}) = \sigma_0 \kappa \frac{\partial}{\partial q} \int \int \nabla^2 (m-1) q^* \cdot \nabla q^* \, dl \]

\[
= \sigma_0 \kappa \frac{\partial}{\partial q} \int \int \nabla^2 (m-1) q^* \cdot \nabla q^* \, dl \frac{\partial A}{\partial q} \\
= \sigma_0 \kappa \left( \frac{\partial q}{\partial A} \right)^2 \frac{\partial A}{\partial q} \tag{B2} \]

This form of effective diffusivity \( \kappa_{\text{eff}} \) is identical to that in Nakamura and Zhu (2010) (appendix D). Letting \( m = 1 \), \( \kappa_{\text{eff}} \) takes the form of (13) which is identical to that in Nakamura (1996). Note that we could use (B2) to calculate the symmetric mixing term \( \mathcal{M}(\dot{q}) \) since in our model \( \dot{q} \) does take a hyperdiffusion form. However, we choose to directly integrate the small-scale diffusion \( \dot{q} \) to be consistent with our calculation of the asymmetric mixing term \( -q \partial \mathcal{M}(\dot{q}) / \partial q \).

References


Appenzeller, C., and H. C. Davies, 1992: Structure of stratospheric intrusions into the troposphere. *Nature*, 358 (6387), 570–572, URL http://dx.doi.org/10.1038/358570a0.


FIG. 1: Schematics depicting the integration area for the asymmetric mixing (left) and symmetric mixing (right) terms. (a) and (b) illustrate the calculation of total asymmetric and symmetric mixing. (c) illustrates the asymmetric mixing related to Rossby wave breaking (dark green), and (d) illustrates the symmetric mixing by waves (all shading), and the part related to Rossby wave breaking (dark blue shading). See text for additional details.
FIG. 2: Snapshots of the concentration of the stratospheric tracer (shading) on the 335K isentrope in the idealized LC1 (left column) and LC2 (right columns) simulations. Dashed contours denote the tropopause defined as the 0.5 contour of the stratospheric tracer.
\[
\mathcal{M}(q)\bigg|_{t=0}^t = -\int_{t=0}^t q \frac{\partial \mathcal{M}(\dot{q})}{\partial q} \, dt + \int_{t=0}^t \mathcal{M}(\dot{q}) \, dt
\]

**Fig. 3:** Time integration of asymmetric mixing (green) and symmetric mixing (red) of tracer \( T \) and \( S \) for LC1 and LC2 simulations (unit: \( 1 \times 10^{16} \) kg). Dashed blue lines denote the actual change in tracer mass due to mixing, with the solid blue lines denoting that estimated by the sum of the asymmetric and symmetric mixing terms (solid blue = solid red + solid green). Positive (negative) values denote a poleward (equatorward) mixing into the stratosphere (troposphere).
FIG. 4: Climatology of zonal-mean potential temperature $\theta$ in the 25-year climate run (solid contours, unit: K). The dashed line denotes the climatological 2 PVU surface which represents the dynamical tropopause. Pink and light green shading denote the underworld and the middleworld analyzed here.
**Fig. 5:** Upper row: Climatology of eddy diffusivity (shading) as a function of approximate equivalent latitude of the tracer contours for the (a) total, (b) Rossby wave breaking (RWB) contribution. Lower row: The same as the upper row, but for net mass flux due to asymmetric mixing. The two solid black contours in the left column denote the $20\text{m/s}$ and $30\text{m/s}$ contours of the zonal mean zonal wind. Contours in the right column denote the frequency of occurrence of RWB events. The black dashed line is the 2PVU contour that represents the dynamical tropopause.
**Fig. 6:** **Upper row:** Contribution of (a) anticyclonic wave breaking (AWB) contribution, and (b) cyclonic wave breaking (CWB) to eddy diffusivity (shading) as a function of approximate equivalent latitude of the coordinate contours. **Lower row:** The same as the upper row, but for net mass flux due to asymmetric mixing. Solid grey contours denote the frequency of occurrence of AWB in the left column and CWB in the right column, with a contour interval of 0.15. The black dashed line is the 2PVU contour that represents the dynamical tropopause.
**FIG. 7:** **Upper row:** Eddy diffusivity (shading) as a function of approximate equivalent latitude in the idealized lifecycle simulations averaged over days 5-13 for (a) LC1, and (b) LC2. **Lower row:** Climatology of eddy diffusivity as a function of approximate equivalent latitude in the 25-year climate simulation for (c) anticyclonic wave breaking (AWB), and (d) cyclonic wave breaking (CWB), normalized by their respective frequency of occurrence.
Fig. 8: Composite frequency of occurrence of Rossby wave breaking (RWB) as a function of approximate equivalent latitude with respect to various jet positions for (a) anticyclonic wave breaking (AWB), (b) cyclonic wave breaking (CWB), and (c) the total RWB in the underworld (270-295K).
Fig. 9: Composite mass flux (unit: $1 \times 10^{11}$ kg/s) as a function of approximate equivalent latitude in the underworld with respect to various jet positions for (a) the total, (b) Rossby wave breaking (RWB), (c) anticyclonic wave breaking (AWB), and (d) cyclonic wave breaking (CWB).
FIG. 10: Composite eddy diffusivity (unit: $1 \times 10^6 m^2 s^{-1}$) as a function of approximate equivalent latitude in the underworld with respect to various jet positions for (a) the total, (b) Rossby wave breaking (RWB), (c) anticyclonic wave breaking (AWB), and (d) cyclonic wave breaking (CWB).
Fig. 11: Snapshots of stratospheric tracer concentration (shading) in the (a) LC1, and (b) LC2 idealized lifecycle simulations on the day of maximum intensity. The black dashed line denotes the zonal-mean contour of 2PVU averaged over the eddy life cycle.
Fig. 12: Composite frequency of occurrence of Rossby wave breaking as a function of approximate equivalent latitude with respect to various jet positions for (a) anticyclonic wave breaking (AWB), and (b) cyclonic wave breaking (CWB) in the middleworld (300-350K).
FIG. 13: Composite mass flux (unit: $1 \times 10^{10} \text{kg/s}$) across the dynamical tropopause due to asymmetric mixing with respect to various jet positions for (a) the total, (b) Rossby wave breaking (RWB), (c) anticyclonic wave breaking (AWB), and (d) cyclonic wave breaking (CWB).
Fig. 14: Composite eddy diffusivity (unit: $1 \times 10^6 m^2 s^{-1}$) as a function of approximate equivalent latitude along the dynamical tropopause with respect to various jet positions for (a) the total, (b) Rossby wave breaking (RWB), (c) anticyclonic wave breaking (AWB), and (d) cyclonic wave breaking (CWB).